# Math Exam for the 2013 admission process at Nada Junior High School, Kobe
## 1st Day (60 minutes)

Note
- The figures are not necessarily precise.
- Use 3.14 as the circle ratio.
- The volume of a pyramid equals (the area of base)×(height)×$\frac{1}{3}$.

Fill the blank with an appropriate number.

$\boxed{1}$ $(\frac{1}{11}-\frac{1}{183})\div 43=(\frac{1}{\square}-\frac{1}{671})\div 167$ (4 points)

$\boxed{2}$ There's a candy which costs \$1.8 each. A bag of 3 candies costs \$5, and a box of 10 candies costs \$19. One day, they sold 107 candies with revenue of \$199. This day, □ bags were sold. (8 points)

$\boxed{3}$ If we apply the process explained below 100 times to the 8 digit number 12345678, we'll get □.
Process: Move the 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th numbers from the left into the 2nd, 4th, 6th, 8th, 1st, 3rd, 5th, 7th place from the left, respectively. That is, change ABCDEFGH into EAFBGCHD. (8 points)

$\boxed{4}$ Among the irreducible fractions which are greater than 0.5 and smaller than 0.51, there're □ fractions whose denominators are not greater than 100. (8 points)

$\boxed{5}$ Starting with 2×2=4, we keep producing new numbers by multiplication, under the condition that 2 and any produced number can be used anytime.

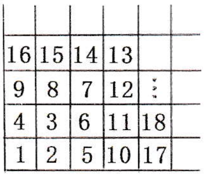
For example, 2×2=4, 4×2=8, 8×2=16 show that 16 is produced in 3 multiplications, while 2×2=4, 4×4=16 show that 16 is produced in 2 multiplications as well.

Under the rule, the minimal number of multiplications needed to produce 512($=2^9$), 32768($=2^{15}$) is ①, ②, respectively. (4 points×2)

$\boxed{6}$ On a straight road connecting towns A and B, there're 4 intersections, which are apart by 230m. Each intersection is equipped with a light, which turns green for 28 seconds then yellow and red for 32 seconds. If we drive from A to B at the constant speed of 11.5 m/s and pass the 1st light at the moment that it turns yellow from green, we'll pass the other 3 lights at the moment that they turn yellow from green. □ m/s is the fastest constant speed at which we drive from B to A without stopping at any light. We assume that a light is green at the moment that it turns green (yellow) from red (green). (8 points)

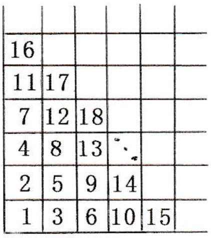
$\boxed{7}$ We have a 2 digit integer AB. The 3 digit integer A0B and the 5 digit integer CACBC are divisible by AB, where A,B,C are nonzero and distinct each other. The 5 digit integer CACBC is □. (8 points)

**8** We fill boxes with integers 1 through 400 under certain rule. First, we did as in the figure 1. Then we erased them and did as in the figure 2. $\square$ boxes were filled with an integer twice. (8 points)



| | | | | |
|---|---|---|---|---|
| | | | | |
| 16 | 15 | 14 | 13 | |
| 9 | 8 | 7 | 12 | ⋮ |
| 4 | 3 | 6 | 11 | 18 |
| 1 | 2 | 5 | 10 | 17 |

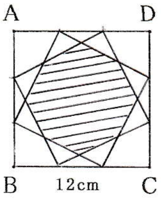
Fig.1



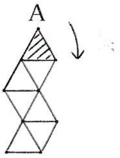
| | | | | | |
|---|---|---|---|---|---|
| | | | | | |
| 16 | | | | | |
| 11 | 17 | | | | |
| 7 | 12 | 18 | | | |
| 4 | 8 | 13 | ⋱ | | |
| 2 | 5 | 9 | 14 | | |
| 1 | 3 | 6 | 10 | 15 | |

Fig.2

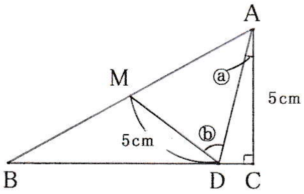
**9** In the figure right, ABCD is a square with side 12cm, and each side is equally divided into 3. The area of the shaded octagon is $\square$ cm$^2$. (8 points)



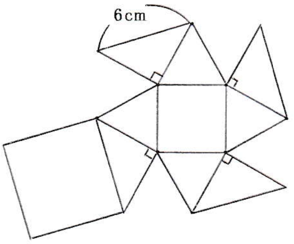
**10** 7 equilateral triangles with side 3cm are placed as in the right. If the shaded one moves around the others without sliding in the direction indicated until it comes back to the original position, the length of the curve traced by the vertex A is $\square$ cm. Note that the vertex A doesn't necessarily go back to the original position. (8 points)



**11** On the right figure, M is the midpoint of the edge AB, ⓐ=15°, AC and MD are 5cm. Then ⓑ= ①°, BD= ② cm. (4 points × 2)



**12** The volume of the solid built from the right exploded view, of 2 squares, 4 equilateral triangles and 4 isosceles right triangles, is $\square$ cm$^3$. (8 points)



**13** A cube shaped container is fixed inclined. Pouring water in it, we get the figure 1. Pouring more, we get the figure 2, and the water occupies $\frac{11}{14}$ of the cube. The water occupies $\square$ of the cube in the figure 1. (8 points)

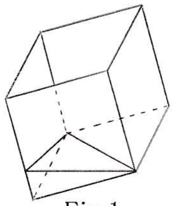


Fig.1

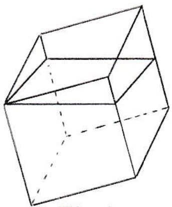


Fig.2

# 2nd Day (60 minutes)

Note
- Just give the answer for $\boxed{5}$ (1),(2),(3).
  Explain how you've got the answer for the other questions.
- The volume of a pyramid equals (the area of base)×(height)×$\frac{1}{3}$.

$\boxed{1}$ Points ***P***,***Q*** are 42km apart, and two boys ***A***,***B*** left ***P*** for ***Q*** at the same moment. ***A*** kept a constant speed, while ***B*** walked to the point ***M***, which is 28km from ***P***, at a constant speed, rested at ***M*** for 20 minutes, then walked to ***Q*** at $\frac{1}{3}$ of the original speed. ***A*** passed by ***B*** 1 hour and 21 minutes after ***B*** left ***M***, and ***A*** arrived at ***Q*** 20 minutes before ***B*** did. (8 points×2)

(1) When ***B*** arrived at ***M***, how many kilometers was ***A*** behind of ***B***?

(2) How long did it take for ***A*** to arrive at ***Q*** since the departure from ***P***?

$\boxed{2}$ 2013 is made of 4 consecutive numbers 0,1,2,3. 4213 is also made of 4 consecutive numbers 1,2,3,4. We consider such 4 digit numbers made of 4 consecutive numbers. (8 points×2)

(1) How many of them are divisible by 3?

(2) How many such numbers are there that its leftmost 3 digit and the rightmost 1 digit have the same remainder when they're divided by 3?

$\boxed{3}$ The figures 1,2 indicate that rectangles ABCD, PQRS are packed only with squares. (8 points×2)

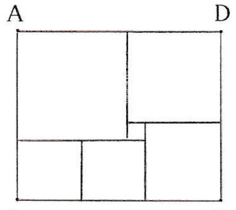


Fig.1

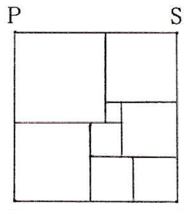


Fig.2

(1) Express AB:AD, the ratio of lengths, in the simplest form.

(2) Express PQ:PS, the ratio of lengths, in the simplest form.

$\boxed{4}$ (1) On the figure 1, find the length of AB,AC. (8 points)

(2) In the figure 2, a circle, centered at D with radius DE, intersects another, with diameter EF, at H in the rectangle DEFG. (8 points×2)

① Find the length of EH.

② Find the area of the quadrilateral EFGH.

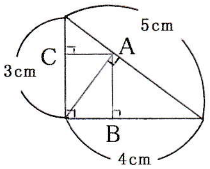


Fig.1

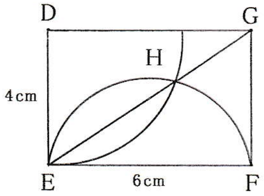


Fig.2

$\boxed{5}$ Figure 1 is a cube with side 6cm built of 36 blocks with sides 1cm, 2cm, 3cm. (7 points×4)

(1) We cut the cube with the plane through A,B,C, and leave the part containing the vertex D. Write the joints between the blocks on the section in the figure 2.

(2) We cut the figure 2 with the plane through D,B,E and leave the part containing the vertex C. Write the joints between the blocks on the section in the figure 3.

(3) The figure 3 consists of □ blocks, among which □ blocks are in the original shape.

(4) Find the minimum of volumes of the blocks contained in the figure 3.

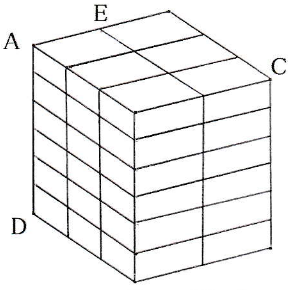


Fig.1

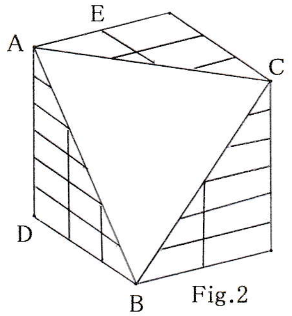


Fig.2

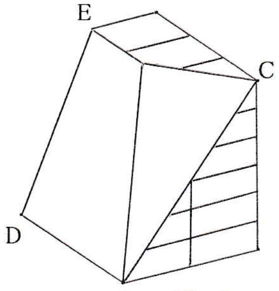


Fig.3

# Solution for the 2013 Math Exam at Nada Junior High School, Kobe
## 1st Day

$\boxed{1}$ $(\frac{1}{11}-\frac{1}{183})\div 43=\frac{183-11}{2013}\div 43=\frac{4}{2013}$
$=\frac{668}{2013}\div 167.$ $\frac{1}{671}+\frac{668}{2013}=\frac{1}{\boxed{3}}.$

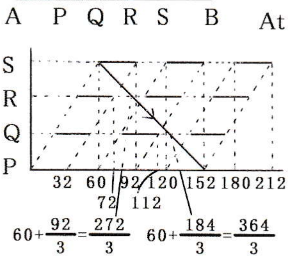
$\boxed{2}$ Suppose that they sold $x$ units, $y$ bags and $z$ boxes of candies. Then
$1.8x+5y+19z=199 \Leftrightarrow 9x+25y+95z=995$ …①
$x+3y+10z=107$ …②
①−②×9 gives us $-2y+5z=32 \Leftrightarrow y=\frac{5}{2}z-16$ …③
$\therefore x \underset{②}{=} -3y-10z+107 \underset{③}{=} -\frac{35}{2}z+155.$
As $x\geqq 0$ and $y\geqq 0$, $\frac{32}{5}\leqq z\leqq\frac{62}{7}$. Furthermore, $z$ is even because $x,y$ are integers. Hence $z=8$.
$\therefore y \underset{③}{=} \frac{5}{2}\times 8-16=\boxed{4}.$

$\boxed{3}$ 12345678→51627384→75318642→87654321→48372615→24681357→12345678 indicates that the 8 digit integer returns to itself in 6 operations. Hence 100=6×16+4 operations yield $\boxed{48372615}$, which is obtained by 4 operations.

$\boxed{4}$ Case (i) Denominator is even.
If integers $m,n$ satisfy $0.5<\frac{n}{2m}<0.51$
$\Leftrightarrow m<n<1.02m$, we have $1.02m>m+1 \Leftrightarrow m>50$.
Then $2m>100$, that is, there's no fraction whose denominator is not greater than 100 between 0.5 and 0.51.
Case (ii) Denominator is odd.
The denominator is not less than 3. Note that any integer $m$ satisfies $\frac{m}{2m+1}<0.5<\frac{m+1}{2m+1}$.
$\frac{m+1}{2m+1}<0.51$ holds iff $1.02m+0.51>m+1$
$\Leftrightarrow m>24.5$. Hence the cases $m=25,\cdots,49$ produce fractions $\frac{26}{51},\cdots,\frac{50}{99}$, which are greater than 0.5 and smaller than 0.51. Furthermore, they are irreducible because $2(m+1)-(2m+1)=1$; the left hand side is divisible by any common divisor of $m+1$ and $2m+1$ so that 1 is the only such.
$\frac{m+2}{2m+1}<0.51$ holds iff $m+2<1.02m+0.51$
$\Leftrightarrow m>74.5$, which implies $2m+1>100$.
Finally, the $\boxed{25}$ fractions listed above satisfy the required condition.

$\boxed{5}$ In order to produce a number as big as possible, we should do 2×2=4, 4×4=16, 16×16=256, $256\times 256=2^{16}$, squaring the number just obtained.
Thus it isn't possible to produce 512 in 3 times, but is in $\underset{①}{\boxed{4}}$ times. (2×2=4, 4×4=16, 16×16=256, 256×2=512.)
We can produce $2^{15}$ in $\underset{②}{\boxed{5}}$ times: 2×2=4, 4×2=8, 8×8=64, $64\times 64=2^{12}$, $2^{12}\times 8=2^{15}$.
If it were possible to produce it in 4 times, the 4th multiplication would be $2^{14}\times 2$, $2^{13}\times 2^2$, $2^{12}\times 2^3$, $2^{11}\times 2^4$, $2^{10}\times 2^5$, $2^9\times 2^6$ or $2^8\times 2^7$.
However, $2^8$ is the maximum which is produced by the 3rd multiplication, and we cannot produce both $2^8$ and $2^7$ in 3 multiplications.

$\boxed{6}$



Call the intersections P,Q,R,S as in the figure. At the speed of 11.5m/s, it takes 230÷11.5=20 seconds between adjacent intersections so that the green light period varies by 20 seconds as the diagram shows. (— means yellow and red.)
↘ indicates the case in which we drive from B to A, without stopping at any light, at the fastest constant speed of 690÷92=$\boxed{7.5}$ m/s.

$\boxed{7}$ As A0B=100A+B is a multiple of AB=10A+B, (100A+B)−(10A+B)=90A is also a multiple of 10A+B, that is, 10A+B is a divisor of 90,180,⋯ or 90×9.
Also, CACBC=10101×C+1000A+10B =10101C+10(100A+B) is a multiple of 10A+B so that 10101C is a multiple of 10A+B as well.
Because 10101=3×7×13×37 and 10A+B is a 2 digit common divisor of 90A and 10101C, there'll be the following cases:
If A=1, (AB,C)=(15,$\underset{\uparrow}{5}$),(18,6)
B≠C is not satisfied.
If A=2, 20 is the only divisor of 90A=180 between 20 and 29. However, B≠0 is not satisfied.
Quite similarly, if A=3,4,5,6,7,8,9, there's no (AB,C) satisfying the condition.
Thus we get (A,B,C)=(1,8,6). ∴CACBC=$\boxed{61686}$.

$\boxed{8}$

Fig.1

| | Col 1 | Col 2 | Col 3 | ⋯ | Col 20 |
|---|---|---|---|---|---|
| Row 20 | 400 | 399 | 398 | ⋯ | 381 |
| | ⋮ | ⋮ | ⋮ | ⋱ | ⋮ |
| Row 3 | 9 | 8 | 7 | ⋯ | 364 |
| Row 2 | 4 | 3 | 6 | ⋯ | 363 |
| Row 1 | 1 | 2 | 5 | ⋯ | 362 |

Fig.2

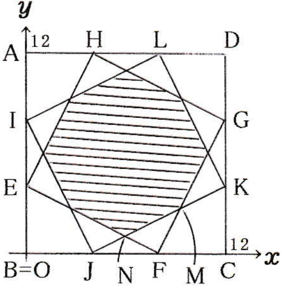
| | Col 1 | Col 2 | Col 3 | Col 4 |
|---|---|---|---|---|
| | | | | |
| Row 4 | 7 | | | |
| Row 3 | 4 | 8 | | |
| Row 2 | 2 | 5 | 9 | |
| Row 1 | 1 | 3 | 6 | 10 |

In the figure 1, an integer is written in a box which is located in the 20th row or lower and 20th column or lefter.

In the figure 2, the box of the 1st row, $n$th column is filled with $1+2+\cdots+n=\dfrac{n(n+1)}{2}$. Hence the box of the 1st row, 28th column corresponds to $\dfrac{28\times29}{2}=406$. Therefore, there're $1+2+\cdots+8=36$ filled boxes in the 21st-28th rows, and $36-\underset{\substack{\uparrow\\ 401-406}}{6}=30$ in the 21st-27th columns.

Thus $400-(36+30)=\boxed{334}$ boxes were filled with an integer twice.

$\boxed{9}$



Take the coordinate axes and name the vertices, as indicated.

Then EFGH,IJKL are squares with side $\sqrt{4^2+8^2}=4\sqrt{5}$ cm and area 80cm$^2$.

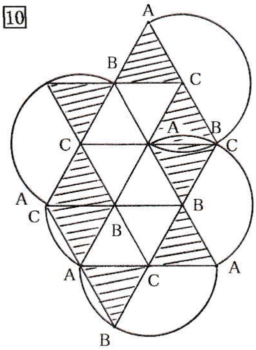
The lines FG: $y=2(x-8)$ and JK: $y=\dfrac{1}{2}(x-4)$ intersect in $M(\dfrac{28}{3},\dfrac{8}{3})$. $\therefore MF=\sqrt{5}(\dfrac{28}{3}-8)=\dfrac{4\sqrt{5}}{3}$.

The lines JK and EF: $y=-\dfrac{1}{2}x+4$ intersect in N(6,1).

$\therefore NF=\sqrt{5}$, $\triangle FMN=\dfrac{1}{2}\times\dfrac{4\sqrt{5}}{3}\times\sqrt{5}=\dfrac{10}{3}$.

The area to find is $\square EFGH-4\triangle FMN=\boxed{\dfrac{200}{3}}$ cm$^2$.

$\boxed{10}$



The vertex A traces the arcs shown, each of which has radius 3cm. The sum of their central angles is 240°+180°×3+60°=840°.

Hence the length to find is $3\times2\times3.14\times\dfrac{840}{360}=\boxed{43.96}$ cm.

$\boxed{11}$ Let N be the midpoint of BC. It follows from the midsegment theorem that $MN=\dfrac{AC}{2}=\dfrac{5}{2}$ cm, $\angle MNB=\angle ACB=90^\circ$

$\therefore \angle MDN=30^\circ$, $DN=\dfrac{5}{2}\sqrt{3}$.

$\therefore$ ⓑ$=180-\angle ADC-\angle MDN$

$=180^\circ-(90^\circ-15^\circ)-30^\circ=\underset{①}{\boxed{75^\circ}}$

$BD=BN+ND=CN+ND=2ND+CD=5\sqrt{3}+5\tan15^\circ$

$=5\sqrt{3}+5\tan(60^\circ-45^\circ)=5\sqrt{3}+5\cdot\dfrac{\tan60^\circ-\tan45^\circ}{1+\tan60^\circ\tan45^\circ}$

$=5\sqrt{3}+5\cdot\dfrac{\sqrt{3}-1}{1+\sqrt{3}}=5\sqrt{3}+5\cdot\dfrac{4-2\sqrt{3}}{2}=\underset{②}{\boxed{10}}$ cm

$\boxed{12}$

Fig.1

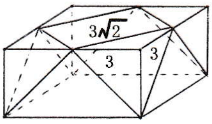
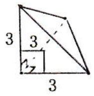


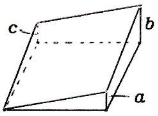
Fig.2



We get the solid shown in the figure 1 : a box with sides 6cm, 6cm, 3cm, minus 4 tetrahedrons shown in the figure 2.

Its volume is $6\times6\times3-\dfrac{1}{2}\times3\times3\times3\times\dfrac{1}{3}\times4=\boxed{90}$ cm$^3$.

$\boxed{13}$



Suppose that the cube has side 1, and let $a,b,c$ be the lengths of sides of dry part of the figure 2, as shown left.

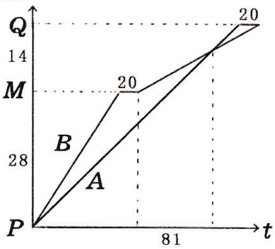
Then it follows from the condition that $a=c$, $b=a+c$, $\dfrac{a+b+c}{4}=1-\dfrac{11}{14}=\dfrac{3}{14}$.

$\therefore a=c=\dfrac{3}{14}$, $b=\dfrac{3}{7}$.

Hence in the figure 1, water occupies a tetrahedron with perpendicular sides 1,1, $b-a=\dfrac{3}{14}$. Its volume is $\dfrac{1}{2}\times\dfrac{3}{14}\times\dfrac{1}{3}=\boxed{\dfrac{1}{28}}$.

# 2nd Day

$\boxed{1}$



(1) Suppose that $A$ walked at $a$ m/min. and $B$ walked from $M$ at $b$ m/min. Then $B$ walked up to $M$ at $3b$ m/min.

The condition tells us that $A,B$ walked for the same period of time, that is,

$$\frac{42000}{a}=\frac{28000}{3b}+\frac{14000}{b}.\ \therefore a=\frac{9}{5}b.$$

When $B$ arrived at $M$, $A$ has walked for $28\times\frac{a}{3b}=16.8$ km so that he was $28-16.8=\underline{11.2}$ km behind of $B$.

(2) Suppose that it took $t$ minutes for $B$ to walk from $P$ to $M$. Then $A$ walked for $a(t+20+81)=28000+81b$ m since the departure from $P$ until he passed by $B$.

As $at=16800$ and $b=\frac{5}{9}a$ by (1), $16800+101a=28000+45a \Leftrightarrow a=200$.

Hence it took $42000\div200=210$ minutes $=\underline{\text{3 hours and 30 minutes}}$ for $A$ to arrive at $Q$.

$\boxed{2}$ (1) 4 consecutive numbers among $0,1,\cdots,9$ are $(0,1,2,3),(1,2,3,4),(2,3,4,5),(3,4,5,6),(4,5,6,7),(5,6,7,8),(6,7,8,9)$, of which $(0,1,2,3),(3,4,5,6),(6,7,8,9)$ are such that their sum is a multiple of 3.

$(0,1,2,3)$ yields $\underset{\substack{\uparrow\\ \text{digit of 1000}}}{3}\times3!=18$ 4 digit integers, and $(3,4,5,6),(6,7,8,9)$ yield $4!=24$ each.

Thus there're $18+24\times2=\underline{66}$ in total.

(2) Case (i) (The rightmost digit)$\equiv0$ mod 3.

◈ The rightmost digit is 0 ⋯ the leftmost 3 digit is a permutation of 1,2,3, and there're $3!=6$.

◈ The rightmost digit is 3 ⋯ the leftmost 3 digit made from (0,1,2) ; there're $\underset{\substack{\uparrow\\ \text{digit of 100}}}{2}\times2!=4$.

made from (4,5,6) ; there're $3!=6$.

◈ The rightmost digit is 6 ⋯ the leftmost 3 digit made from (3,4,5),(7,8,9) ; there're 6 each.

◈ The rightmost digit is 9 ⋯ the leftmost 3 digit is made from (6,7,8), and there're 6.

Case (ii) (The rightmost digit)$\equiv1$ mod 3.

◈ The rightmost digit is 1 ⋯ there's no 4 consecutive numbers satisfying the condition ; $2+3+0\equiv2$, $2+3+4\equiv0$ mod 3.

◈ The rightmost digit is 4 ⋯ the leftmost 3 digit is made from (2,3,5), and there're 6.

◈ The rightmost digit is 7 ⋯ the leftmost 3 digit is made from (5,6,8), and there're 6.

Case (iii) (The rightmost digit)$\equiv2$ mod 3.

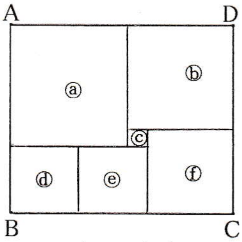
◈ The rightmost digit is 2 ⋯ the leftmost 3 digit is made from (1,3,4), and there're 6.

◈ The rightmost digit is 5 ⋯ the leftmost 3 digit is made from (4,6,7), and there're 6.

◈ The rightmost digit is 8 ⋯ there's no 4 consecutive numbers satisfying the condition.

Summing up above, there're $6\times9+4=\underline{58}$ 4 digit numbers satisfying the condition.

$\boxed{3}$ (1) Let $a,b$ be the length of sides of the squares ⓒ,ⓔ, respectively. Then the length of sides of the other squares are



ⓓ : $b$, ⓕ : $a+b$,

ⓑ : $(a+b)+a=2a+b$,

ⓐ : $(2a+b)+a=3a+b$.

$\therefore AD=(3a+b)+(2a+b)=5a+2b$,

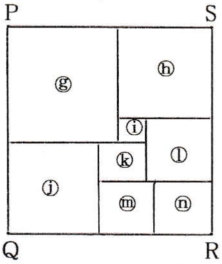
$BC=b+b+(a+b)=a+3b$.

Because AD=BC, $5a+2b=a+3b \Leftrightarrow b=4a$.

$\therefore AB=(3a+b)+b=3a+2b=11a$, $AD=5a+2b=13a$.

$\therefore AB:AD=\underline{11:13}$.

(2) Let $c,d$ be the length of sides of the squares ⓘ,ⓚ, respectively. Then the length of sides of the other squares are



ⓛ : $c+d$,

ⓗ : $c+(c+d)=2c+d$,

ⓜ,ⓝ : $\frac{d+(c+d)}{2}=\frac{c}{2}+d$,

ⓙ : $d+(\frac{c}{2}+d)=\frac{c}{2}+2\mathrm{d}$. As PQ=SR,

ⓖ : $\underset{\text{ⓗ}}{(2c+d)}+\underset{\text{ⓝ}}{(c+d)}+\underset{\text{ⓛ}}{(\frac{c}{2}+d)}-\underset{\text{ⓙ}}{(\frac{c}{2}+2d)}=3c+d$.

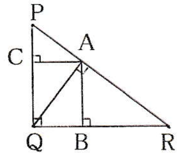
Because PS=QR, $(3c+d)+(2c+d)=(\frac{c}{2}+2d)+(c+2d) \Leftrightarrow d=\frac{7}{4}c$.

$\therefore PQ=(3c+d)+(\frac{c}{2}+2d)=\frac{35}{4}c$,

$PS=(3c+d)+(2c+d)=\frac{17}{2}c$.

$\therefore PQ:PS=\underline{35:34}$.

$\boxed{4}$ (1) Define the vertices P,Q,R as indicated.



Because △RAQ∽△RQP, $\frac{AR}{QR}=\frac{QR}{PR}$. $\therefore AR=\frac{QR^2}{PR}=\frac{16}{5}$.

$\therefore PA=PR-AR=5-\frac{16}{5}=\frac{9}{5}$.

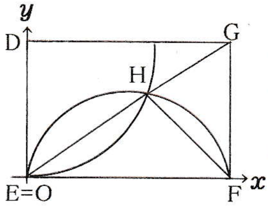
Because △ABR∽△PQR, $\frac{AB}{PQ}=\frac{AR}{PR}$.

$\therefore AB=\frac{AR\cdot PQ}{PR}=\underline{\frac{48}{25}}$ cm.

Because △PCA∽△PQR, $\frac{AC}{RQ}=\frac{PA}{PR}$.

$\therefore AC=\frac{PA\cdot RQ}{PR}=\underline{\frac{36}{25}}$ cm.

(2)① Take the coordinate axes as indicated.



The circle centered at D is given by

$x^2+(y-4)^2=16$ ···(i)

The circle with diameter EF is given by

$(x-3)^2+y^2=9$ ···(ii)

(i)−(ii) : $6x-9-8y+16=7 \Leftrightarrow y=\frac{3}{4}x$ ···(iii)

This is the equation of the line EH. Plugging (iii) into (ii), $(x-3)^2+\frac{9}{16}x^2=9 \Leftrightarrow x=0,\frac{96}{25}$.

This and (iii) tell us $H(\frac{96}{25},\frac{72}{25})$.

$\therefore EH=\sqrt{(\frac{96}{25})^2+(\frac{72}{25})^2}=\underline{\frac{24}{5}}$ cm.

② (Quadrilateral EFGH)=△EFH+△FGH

$=\frac{1}{2}\times6\times\frac{72}{25}+\frac{1}{2}\times4\times(6-\frac{96}{25})=\underline{\frac{324}{25}}$ cm$^2$.

$\boxed{5}$ (1)

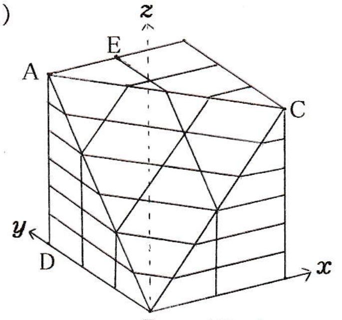


Fig.2

(2)

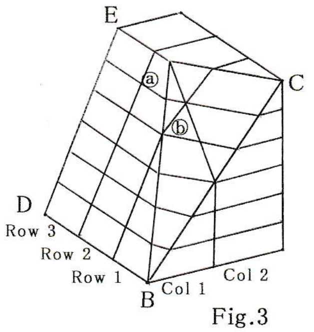


Fig.3

(3) Among the 36 blocks used to build the cube, only the top one of the 1st row, 1st column was cut off completely so that the figure 3 consists of $\boxed{35}$ blocks.

No block in the 1st column has the original shape, because they were cut off by the plane through D,B,E.

On the blocks in the 2nd column, the number of originally shaped ones are 6, 5 (except for the top one), 3 in the 3rd, 2nd, 1st row, respectively.

Thus there're 6+5+3= $\boxed{14}$ in total.

(4) Let ⓐ be the top block of the 2nd row, 1st column, and ⓑ be the 5th block from the bottom of the 1st row, 1st column.

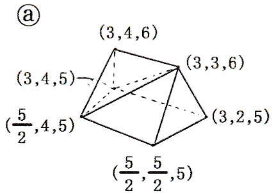
Compare the volume of these.

Take the coordinate axes as in the figure 2.

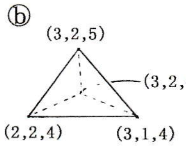
The plane through A(0,6,6), B(0,0,0), C(6,0,6) is $x+y-z=0$, the plane through D(0,6,0), B, E(3,6,6) is $2x-z=0$, and their line of intersection is $y=x$, $z=2x$.

Hence the vertices of ⓐ,ⓑ are

ⓐ



ⓑ



The volume of ⓐ equals the sum of the volume of a pyramid whose base is a trapezoid (upper base $\frac{3}{2}$, lower base 2, height $\frac{1}{2}$) and height is 1 and the volume of a tetrahedron whose base is an isosceles right triangle with equal side 1 and height is $\frac{1}{2}$ :

$\frac{1}{2}\times(\frac{3}{2}+2)\times\frac{1}{2}\times1\times\frac{1}{3}+\frac{1}{2}\times1^2\times\frac{1}{2}\times\frac{1}{3}=\frac{3}{8}$.

The volume of ⓑ equals $\frac{1}{2}\times1^3\times\frac{1}{3}=\frac{1}{6}$.

Therefore, the minimal volume is $\underline{\frac{1}{6}\text{ cm}^3}$ .

# Math Exam for the 2011 admission process at Nada Senior High School, Kobe

Note: Just give the answer for $\boxed{1}$, $\boxed{2}$ (1)(2).
Explain how you've got the answer for the other questions.

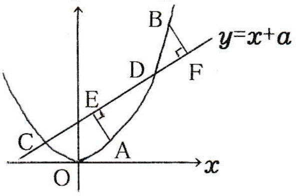
$\boxed{1}$ Fill the blank with an appropriate number.

(1) If $x=1+\sqrt{2}$, $y=2+\sqrt{3}$, $z=4+\sqrt{6}$, $xyz-4xy-yz-2zx+8x+4y+2z-8=$□.

(2) Let $a,b$ be constants different from 0, and $x,y$ be coordinates on a plane. If the two lines $x+\sqrt{6}y=9\sqrt{2}$, $\dfrac{x}{a}+\dfrac{y}{b}=1$ intersect where the two lines $\dfrac{x}{b}+\dfrac{y}{a}=0$, $\sqrt{6}x+y=8\sqrt{3}$ do, then $a=$□, $b=$□.

(3) Let $a$ be a positive odd integer, and $b$ be a prime. If the quadratic equation $x^2-ax-b^3=0$ has a root which is an integer, then $a=$□, $b=$□.

(4) Let A(1,1), B(3,9), points on the parabola $y=x^2$. If $\ell$ : $y=x+a$ ($a$ is a constant) is the line intersecting the parabola in two points C,D as indicated on the figure, and AE=BF, where AE,BF are perpendicular to $\ell$, then $a=$□ and CD=□.



$\boxed{2}$ A boy is on the bottom of stairs. Tossing a dice, he goes up 1 step if 1 shows up, 2 steps if 2 or 3 shows up, and goes down 1 step if 4,5 or 6 shows up. In addition, he stays on the bottom if he is there and 4,5 or 6 shows up.

(1) If he is on the 4th step after tossing the dice 3 times, there're □ possible ways in which the number of the dice shows up in the 3 tossings.

(2) If he is on the 2nd step after tossing the dice 3 times, there're □ possible ways in which the number of the dice shows up in the 3 tossings.

(3) Suppose that he is on the 3rd step after tossing the dice 4 times. Find the number of possible ways in which the number of the dice shows up in the 4 tossings.

$\boxed{3}$ There're 2 towns P and Q along a river, where Q is 37.8km downstream of P. A boat left P for Q, and a cruiser left Q for P at the same moment. They passed each other 42 minutes later, and the boat arrived at Q 12 minutes thereafter. Having rested for $x$ minutes at Q, the boat left for P and passed by the cruiser. The time needed for the boat to pass by the cruiser since the departure from Q was a half of the time since the departure from P until the departure from Q. Let $a,b$ and $c$ [m/minute] be the speed of river current, boat and cruiser (on the water without current), respectively.

(1) How long did it take for the cruiser to arrive at P after leaving from Q?

(2) Find $a,b,c$.

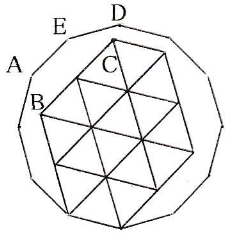
(3) The cruiser arrived at P 7 minutes later than the boat did. Find $x$.

$\boxed{4}$ 16 equilateral triangles with side 1 are placed inside a regular dodecagon with side 1, as the figure indicates.
Let A,B,C,D,E be vertices as shown.

(1) Find the distance between A and B.

(2) Find the distance between C and D.

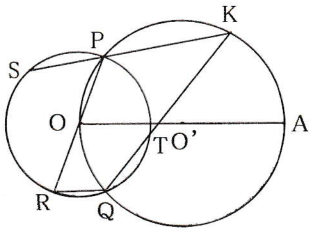
(3) Find the area of the pentagon ABCDE.



$\boxed{5}$ As the figure indicates, two circles O,O’ centered at O,O’ intersect in two points P,Q. O is on the circle O’, and OA is a diameter of the circle O’. K is on the arc PA (the one not containing O and excluding its end points) and S,T,R are on the circle O which are also on the line KP,KQ,OP, respectively.

(1) Prove that PS=QT.

(2) Prove that the line KQ passes through O’ if QR=QT.

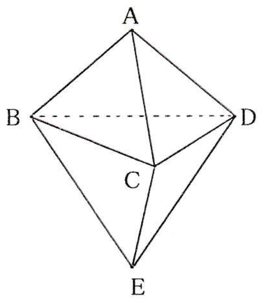


$\boxed{6}$ The figure ABCDE shown right has 6 faces of equilateral triangles with side 6.

(1) Find the distance between A and E.

(2) Find the volume of the tetrahedron ABCE.

(3) Let P,Q,R be the midpoints of AB,BC,CE, respectively. Find the area of the section of the tetrahedron ABCE by the plane through P,Q,R.



# Solution for the 2011 Math Exam at Nada Senior High School, Kobe

$\boxed{1}$ (1) If $x=1+\sqrt{2}$, $y=2+\sqrt{3}$, $z=4+\sqrt{6}$,

$$xyz-4xy-yz-2zx+8x+4y+2z-8$$
$$=(yz-4y-2z+8)x-(yz-4y-2z+8)$$
$$=(x-1)(y-2)(z-4)=\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{6}=\underline{6}.$$

(2) It follows from the condition that the following system of equations in $x$ and $y$ has a root:

$x+\sqrt{6}y=9\sqrt{2}$ ···①, $\dfrac{x}{a}+\dfrac{y}{b}=1$ ···②, $\dfrac{x}{b}+\dfrac{y}{a}=0$ ···③, $\sqrt{6}x+y=8\sqrt{3}$ ···④. ①×$\sqrt{6}$−④ gives us $5y=10\sqrt{3} \Leftrightarrow y=2\sqrt{3}$. $\therefore x \underset{①}{=} 9\sqrt{2}-\sqrt{6}\cdot2\sqrt{3}=3\sqrt{2}$.

Plugging these into ② and ③, we get

$$\frac{3\sqrt{2}}{a}+\frac{2\sqrt{3}}{b}=1 \text{ ···②'},\quad \frac{3\sqrt{2}}{b}+\frac{2\sqrt{3}}{a}=0 \text{ ···③'}$$

$$\text{②'}\times\sqrt{3}-\text{③'}\times\sqrt{2}:\ \frac{\sqrt{6}}{a}=\sqrt{3} \Leftrightarrow a=\underline{\sqrt{2}}.$$

$$\text{②'}\times\sqrt{2}-\text{③'}\times\sqrt{3}:\ -\frac{\sqrt{6}}{b}=\sqrt{2} \Leftrightarrow b=\underline{-\sqrt{3}}.$$

(3) If $x^2-ax-b^3=0$ has a root $m$ which is an integer, the other root is $a-m$, which is also an integer. Because $a$ is odd, their parity is different so that their product $-b^3$ is even. As $b$ is a prime, $b=\underline{2}$. Hence $\{m,a-m\}=\{-1,8\}$ or $\{1,-8\}$. As $a=m+(a-m)$ is positive, $\{m,a-m\}=\{-1,8\}$ so that $a=\underline{7}$.

(4) $\mathrm{AE}=\dfrac{|1-1+a|}{\sqrt{2}}=\dfrac{|a|}{\sqrt{2}}$, $\mathrm{BF}=\dfrac{|3-9+a|}{\sqrt{2}}=\dfrac{|a-6|}{\sqrt{2}}$.

$\therefore \mathrm{AE}=\mathrm{BF} \Leftrightarrow |a|=|a-6| \Leftrightarrow a=\underline{3}$.

The $x$-coordinates of C,D are $x^2=x+3$ $\Leftrightarrow x=\dfrac{1\pm\sqrt{13}}{2}$. As the slope of CD is 1,

$$\mathrm{CD}=\sqrt{2}\left(\frac{1+\sqrt{13}}{2}-\frac{1-\sqrt{13}}{2}\right)=\underline{\sqrt{26}}.$$

$\boxed{2}$ If he tosses a dice once, he will go up 1step, 2 steps, go down 1 step, or stay on the bottom. We express these results as +1,+2,−1,0, respectively.

(1) If he is on the 4th step after tossing the dice 3 times, one of 0+2+2,1+1+2,1+2+1 or 2+1+1 has happened. 0+2+2 can happen in 3×2×2=12 ways, and the others can happen in 1×1×2=2 ways. Hence there're $12+2\times3=\underline{18}$ possible ways.

(2) If he is on the 2nd step after tossing the dice 3 times, one of 0+0+2,0+1+1,1−1+2,1+2−1,2+1−1 or 2−1+1 has happened.

0+0+2 can happen in 3×3×2=18 ways,

0+1+1 can happen in 3×1×1=3 ways, and

the others can happen in 1×3×2=6 ways.

Hence there're $18+3+6\times4=\underline{45}$ possible ways.

(3) If he is on the 3rd step after tossing the dice 4 times, one of the following has happened:

| | Location after the 3rd toss | 4th toss |
|---|---|---|
| (i) | 1st step | +2 |
| (ii) | 2nd step | +1 |
| (iii) | 4th step | −1 |

Because of (1) and (2), (ii) and (iii) can happen in 45×1, 18×3 ways, respectively. Furthermore, if he is on the 1st step after the 3rd toss, one of 0+0+1,0+2−1,1−1+1 or 1+1−1 has happened.

0+0+1 can happen in 3×3×1=9 ways,

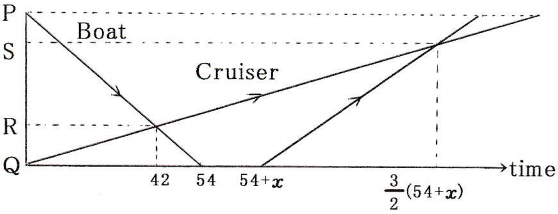
0+2−1 can happen in 3×2×3=18 ways, and

the others can happen in 1×3×1=3 ways.

Hence (i) can happen in (9+18+3×2)×2=66 ways.

In total, the number of possible ways is $45+54+66=\underline{165}$.

$\boxed{3}$ The condition gives us the diagram below:



(1) If R is the place where the boat and the cruiser passed each other, PR:RQ=42:12=7:2. Hence the cruiser proceeded from Q to P in $42\times\dfrac{2+7}{2}=\underline{189}$ minutes.

(2) The boat goes downstream at $(a+b)$m/min, and the cruiser goes upstream at $(c-a)$m/min. Hence the distance between P and Q is

$54(a+b)=189(c-a)=37800$

$\therefore a+b=700$ ···①, $c-a=200$ ···②

If S is the place where the boat passed by the cruiser, the distance between Q and S is

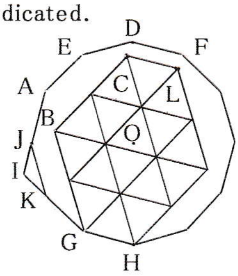
$(b-a)\dfrac{54+x}{2}=(c-a)\dfrac{3(54+x)}{2}$. $\therefore b-a=3(c-a)$ ···③

Plugging ①$\Leftrightarrow b=700-a$···①', ②$\Leftrightarrow c=200+a$···②' into ③$\Leftrightarrow b-3c=-2a$, we get

$700-a-3(200+a)=-2a \Leftrightarrow a=\underline{50},\ b\underset{①'}{=}\underline{650},\ c\underset{②'}{=}\underline{250}.$

(3) It took 189−7=182 minutes for the boat to return to P. As it proceeded from Q to P in $37800\div(b-a)=63$ minutes, $x=182-54-63=\underline{65}$.

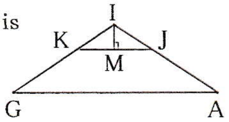
$\boxed{4}$ Define the vertices as indicated.



(1) An interior angle of a regular dodecagon is $\dfrac{180\times(12-2)}{12}=150°$

$\therefore \angle BGK=150-60\times 2=30°$.

As $\angle IKJ=180°-\angle JKG=30°$, BG and JK are parallel and A,B,G are collinear.



As $\angle IJK=30°$ as well, $\triangle IJK$ is isosceles with IJ=IK so that $IM\perp JK$, where M is the midpoint of JK.

$IK\cos 30°=KM=\dfrac{1}{2}$ implies $IK=\dfrac{2}{\sqrt{3}}\cdot\dfrac{1}{2}=\dfrac{1}{\sqrt{3}}$.

$\triangle IJK \backsim \triangle IAG$ gives us $\dfrac{AG}{JK}=\dfrac{IG}{IK}=1+\sqrt{3}$.

$\therefore AG=1+\sqrt{3}$, $AB=AG-BG=\underline{\sqrt{3}-1}$.

(2) Let O indicate the circumcircle and its center of the regular dodecagon. DH,FG are diameters of the circle O so that DGHF is a rectangle, and $\angle GDH=$(an inscribed angle for $\overset{\frown}{GH}$)$=15°$. Hence

$DG=\dfrac{GH}{\tan 15°}=\dfrac{1}{\tan(60°-45°)}=\dfrac{1+\tan 60°\tan 45°}{\tan 60°-\tan 45°}=2+\sqrt{3}$.

C is on DG, and CG=(the height of an equilateral triangle with side 1)$\times 4=2\sqrt{3}$.

$\therefore CD=(2+\sqrt{3})-2\sqrt{3}=\underline{2-\sqrt{3}}$.

(3) As the figure is symmetric with respect to the perpendicular bisector of GH,

$\boldsymbol{S}$:=(pentagon ABCDE)$=\dfrac{1}{2}\{$(regular dodecagon)
 −(equilateral triangle with side 1)$\times 16$
 −(rectangle CDFL)−(quadrilateral AJKG)$\times 2\}$.

As $\triangle OGH=\dfrac{1}{2}GH\cdot\dfrac{DG}{2}=\dfrac{2+\sqrt{3}}{4}$,

(regular dodecagon)$=12\triangle OGH=3(2+\sqrt{3})$.

(Equilateral triangle with side 1)$=\dfrac{1}{2}\cdot 1\cdot\dfrac{\sqrt{3}}{2}=\dfrac{\sqrt{3}}{4}$,

and (rectangle CDFL)$=CD\cdot DF=2-\sqrt{3}$.

The quadrilateral AJKG is a trapezoid with JK=1, $AG=1+\sqrt{3}$ and the height $KG\sin 30°=\dfrac{1}{2}$ so that its area is $\dfrac{1}{2}\{1+(1+\sqrt{3})\}\cdot\dfrac{1}{2}=\dfrac{2+\sqrt{3}}{4}$.

$\therefore \boldsymbol{S}=\dfrac{1}{2}\{3(2+\sqrt{3})-16\cdot\dfrac{\sqrt{3}}{4}-(2-\sqrt{3})-\dfrac{2+\sqrt{3}}{2}\}=\underline{\dfrac{6-\sqrt{3}}{4}}$.

$\boxed{5}$ (1) $\triangle OPS$ and $\triangle OQT$ satisfy

OP=OS=OQ=OT , and

$\angle OPS=180°-\angle OPK=\angle OQT$ ($\because$ the quadrilateral OPKQ is inscribed in the circle O'.)

$\therefore \angle POS=180°-2\angle OPS=180°-2\angle OQT=\angle QOT$.

$\therefore \triangle OPS\equiv\triangle OQT$ (SAS). $\therefore$ PS=QT.

(2) It suffices to prove that $\angle KPQ=90°$ if QR=QT.

If QR=QT, (1) gives us QR=PS so that $\triangle ORQ\equiv\triangle OPS$ (SSS) $\therefore \angle ORQ=\angle OPS \Leftrightarrow SK\parallel RQ$

($\because$ a pair of alternate interior angles are equal.)

$\therefore \angle KPQ=\angle PQR=90°$ ($\because$ PR is a diameter.)

$\boxed{6}$ (1) If AH is the perpendicular from A to the plane BCD, the right triangles AHB,AHC,AHD are congruent for equal hypotenuses (AB=AC=AD) and legs. $\therefore$ BH=CH=DH, that is, H is the circumcenter of $\triangle BCD$. Similarly, $EH\perp BCD$, and H is on AE.

As $\angle BHC=2\angle BDC=120°$, $\triangle BHC$ is similar to $\triangle KIJ$ in $\boxed{4}$ so that $\dfrac{BH}{BC}=\dfrac{KI}{KJ}=\dfrac{1}{\sqrt{3}}$. $\therefore BH=2\sqrt{3}$.

Pythagorean theorem gives $AH=\sqrt{AB^2-BH^2}$ $=2\sqrt{6}$, and $EH=2\sqrt{6}$ as well. $\therefore AE=2\sqrt{6}\times 2=\underline{4\sqrt{6}}$.

(2) It follows from (1) that (tetrahedron ABCE)

=(tetrahedron ABCH)+(tetrahedron EBCH)

$=\dfrac{1}{3}\times\triangle BHC\times AH\times 2=\dfrac{1}{3}\times\triangle BHC\times AE$.

As the distance from H to BC is $BH\sin 30°=\sqrt{3}$, $\triangle BHC=\dfrac{1}{2}\times 6\times\sqrt{3}=3\sqrt{3}$ and the volume to find is $\dfrac{1}{3}\times 3\sqrt{3}\times 4\sqrt{6}=\underline{12\sqrt{2}}$.

(3) Let $\overrightarrow{BA}=\vec{a}$, $\overrightarrow{BC}=\vec{c}$ and $\overrightarrow{BE}=\vec{e}$. Then $\overrightarrow{BQ}=\dfrac{\vec{c}}{2}$.

By the Midsegment theorem, $\overrightarrow{QR}=\dfrac{\overrightarrow{BE}}{2}=\dfrac{\vec{e}}{2}$,

$\overrightarrow{QP}=\dfrac{\overrightarrow{CA}}{2}=\dfrac{\overrightarrow{BA}-\overrightarrow{BC}}{2}=\dfrac{\vec{a}-\vec{c}}{2}$.

Let X be the point of intersection of the plane PQR and the line AE. As X is on AE, we can express $\overrightarrow{BX}=(1-t)\overrightarrow{BA}+t\overrightarrow{BE}=(1-t)\vec{a}+t\vec{e}$ ⋯①, where $t$ is a real. As X is also on the plane PQR, we can express $\overrightarrow{QX}=m\overrightarrow{QP}+n\overrightarrow{QR}=\dfrac{m}{2}(\vec{a}-\vec{c})+\dfrac{n}{2}\vec{e}$ so that $\overrightarrow{BX}=\overrightarrow{BQ}+\overrightarrow{QX}=\dfrac{m}{2}\vec{a}+\dfrac{1-m}{2}\vec{c}+\dfrac{n}{2}\vec{e}$ ⋯②, where $m,n$ are reals.

Because $\vec{a}$, $\vec{c}$, $\vec{e}$ are linearly independent, we get $1-t=\dfrac{m}{2}$, $\dfrac{1-m}{2}=0$, $t=\dfrac{n}{2}$ by comparing the right hand sides of ①,②. Solving these, $t=\dfrac{1}{2}$, $m=n=1$, that is, X is the midpoint H of AE and PQRH is a parallelogram ($\because \overrightarrow{QH}=\overrightarrow{QP}+\overrightarrow{QR}$.) We have

$|\overrightarrow{QP}|=\dfrac{CA}{2}=3$, $|\overrightarrow{QR}|=\dfrac{BE}{2}=3$, $\overrightarrow{QP}\cdot\overrightarrow{QR}=\dfrac{\vec{a}\cdot\vec{e}-\vec{c}\cdot\vec{e}}{4}$,

$\vec{c}\cdot\vec{e}=6^2\cos 60°=18$. Furthermore, (1) and

$|\overrightarrow{AE}|^2=|\vec{e}-\vec{a}|^2=|\vec{e}|^2-2\vec{a}\cdot\vec{e}+|\vec{a}|^2=72-2\vec{a}\cdot\vec{e}$ give

$\vec{a}\cdot\vec{e}=\dfrac{72-(4\sqrt{6})^2}{2}=-12$. $\therefore \overrightarrow{QP}\cdot\overrightarrow{QR}=\dfrac{-12-18}{4}=-\dfrac{15}{2}$.

Therefore, the area to find is (▱PQRH)

$=\sqrt{|\overrightarrow{QP}|^2|\overrightarrow{QR}|^2-(\overrightarrow{QP}\cdot\overrightarrow{QR})^2}=\sqrt{9^2-(-\dfrac{15}{2})^2}=\underline{\dfrac{3\sqrt{11}}{2}}$.

# Math Exam given for the 2025 general admission process at Univ. of Tokyo

## Problems for those who wish to major in Science, Engineering, etc. (150 min.)

$\boxed{1}$ Let A(0,0), B(0,1), C(1,1), D(1,0) be points on a coordinate plane. Let $t$ satisfy $0<t<1$, $P_t, Q_t, R_t$ be the points on the segments AB,BC,CD, respectively, such that $\frac{AP_t}{P_tB}=\frac{BQ_t}{Q_tC}=\frac{CR_t}{R_tD}=\frac{t}{1-t}$, $S_t, T_t$ be the points on the segments $P_tQ_t, Q_tR_t$, respectively, such that $\frac{P_tS_t}{S_tQ_t}=\frac{Q_tT_t}{T_tR_t}=\frac{t}{1-t}$, and $U_t$ be the point on the segment $S_tT_t$ such that $\frac{S_tU_t}{U_tT_t}=\frac{t}{1-t}$.
Furthermore, let A,D be $U_0, U_1$, respectively.

(1) Find the coordinates of the point $U_t$.

(2) Find the area of the domain surrounded by the segment AD and the curve traced by the point $U_t$, $0\leqq t\leqq 1$.

(3) Let $a$ satisfy $0<a<1$. Express the length of the curve traced by the point $U_t$, $0\leqq t\leqq a$, as a polynomial in $a$.

$\boxed{2}$ (1) Prove $\ln x\leqq x-1$ for $x>0$. (2) Find $\lim\limits_{n\to\infty} n\displaystyle\int_1^2 \ln\left(\frac{1+x^{\frac{1}{n}}}{2}\right)dx$.

$\boxed{3}$ A parallelogram ABCD satisfies $\angle ABC=\frac{\pi}{6}$, AB=$a$, BC=$b$, and $a\leqq b$.
Consider a rectangle with the condition:

The vertices A,B,C,D lie on the edges EF,FG,GH,HE, respectively, where an edge includes its ends.

Let $S$ be the area of the rectangle EFGH.

(1) Express $S$ in terms of $a, b$ and $\theta=\angle BCG$.

(2) Express the maximum of $S$ in terms of $a$ and $b$.

$\boxed{4}$ A square number is the square of a nonnegative integer.
Let $a$ be a positive integer, and $f_a(x)=x^2+x-a$.

(1) Let $n$ be a positive integer. Prove that $n\leqq a$, if $f_a(n)$ is a square number.

(2) Denote by $N_a$ the number of positive integers $n$ such that $f_a(n)$ is a square number.
Prove that the conditions (i),(ii) below are equivalent:
(i) $N_a=1$ (ii) $4a+1$ is a prime.

$\boxed{5}$ There're $n(\geqq 2)$ cards numbered 1 through $n$, and we arrange them in a row.
Consider the following operation $(T_i)$, where $i=1,2,\cdots$, or $n-1$.

$(T_i)$ If the number of the $i$th card (from the left end) is greater than that of the $(i+1)$th one, we switch these 2 cards. Otherwise, we do nothing.

Suppose that the number of the $i$th card is $A_i$ $(1\leqq i\leqq n)$ in the beginning, and it turns $i$ for $i=1,\cdots,n$ by $(n-1)$ operations $(T_1),(T_2),\cdots,(T_{n-1})$ followed by $(n-1)$ operations $(T_{n-1}),\cdots,(T_2),(T_1)$.

(1) Prove that at least one of $A_1, A_2$ is not greater than 2.

(2) Let $C_n$ be the number of possible arrangement $A_1\cdots A_n$.
For $n\geqq 4$, express $C_n$ in terms of $C_{n-1}$ and $C_{n-2}$.

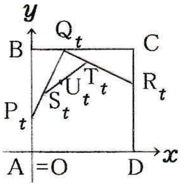
$\boxed{6}$ On a plane of complex numbers, let C be the circle centered at $\frac{1}{2}$ with radius $\frac{1}{2}$, minus zero.

(1) For $z\in C$, prove that the real part of $\frac{1}{z}$ is 1.

(2) If $\alpha, \beta\in C$ and they're distinct, express the domain in which $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ moves around.

(3) If $\gamma$ is a complex number belonging to the complement of the domain in (2), find the maximum and the minimum of the real part of $\frac{1}{\gamma}$.

# Solution for the 2025 Math Exam at Univ. of Tokyo (Science, etc.)



$\boxed{1}$ (1) First, we get $P_t(0,t)$, $Q_t(t,1)$, $R_t(1,1-t)$. Next,

$\overrightarrow{OS_t}=(1-t)\overrightarrow{OP_t}+t\overrightarrow{OQ_t}$

$=(0,t(1-t))+(t^2,t)=(t^2,2t-t^2)$,

$\overrightarrow{OT_t}=(1-t)\overrightarrow{OQ_t}+t\overrightarrow{OR_t}$

$=(t(1-t),1-t)+(t,t(1-t))=(2t-t^2,1-t^2)$,

$\overrightarrow{OU_t}=(1-t)\overrightarrow{OS_t}+t\overrightarrow{OT_t}$

$=(t^2(1-t),(1-t)(2t-t^2))+(t(2t-t^2),t(1-t^2))$

$=(3t^2-2t^3,3t-3t^2)$. $\therefore U_t(3t^2-2t^3,3t-3t^2)$.

(2) Let $(x,y)=(3t^2-2t^3,3t-3t^2)$, $0\leqq t\leqq 1$.

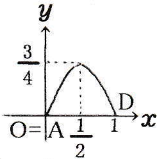
$\dfrac{dx}{dt}=6t-6t^2=6t(1-t)$, $\dfrac{dy}{dt}=3-6t$ give us the table and the graph on the right.

| $t$ | $0$ | | $\frac{1}{2}$ | | $1$ |
|---|---|---|---|---|---|
| $\frac{dx}{dt}$ | | $+$ | | $+$ | |
| $\frac{dy}{dt}$ | | $+$ | $0$ | $-$ | |
| $\binom{x}{y}$ | $\binom{0}{0}$ | ↗ | $\binom{1/2}{3/4}$ | ↘ | $\binom{1}{0}$ |

Thus the area to find is $S=\displaystyle\int_0^1 y\,dx$.

If we put $x=3t^2-2t^3$, we get $y=3t-3t^2$, $dx=(6t-6t^2)dt$, and

| $x$ | $0\to1$ |
|---|---|
| $t$ | $0\to1$ |



$\therefore S=\displaystyle\int_0^1(3t-3t^2)(6t-6t^2)dt$

$=18\displaystyle\int_0^1(t^2-2t^3+t^4)dt=18\left[\frac{t^3}{3}-\frac{t^4}{2}+\frac{t^4}{5}\right]_0^1=\frac{3}{5}$.

(3) The length of the curve is

$$\int_0^a\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\int_0^a\sqrt{36t^2(1-t)^2+(3-6t)^2}\,dt$$

$$=3\int_0^a\sqrt{4t^4-8t^3+8t^2-4t+1}\,dt$$

$$=3\int_0^a(2t^2-2t+1)dt\ \left(\because 2t^2-2t+1=2\left(t-\frac{1}{2}\right)^2+\frac{1}{2}>0\right)$$

$$=[2t^3-3t^2+3t]_0^a=2a^3-3a^2+3a.$$

$\boxed{2}$ (1) Let $f(x)=\ln x-(x-1)$, $x>0$.

Then $f'(x)=\dfrac{1}{x}-1$ gives the table on the right. Hence $f(x)\leqq f(1)=0$, i.e. $\ln x\leqq x-1$.

| $x$ | $(0)$ | | $1$ | |
|---|---|---|---|---|
| $f'(x)$ | | $+$ | $0$ | $-$ |
| $f(x)$ | | ↗ | | ↘ |

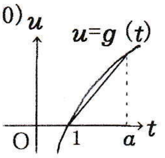
(2) Let $I=\displaystyle\int_1^2\ln\left(\frac{1+x^{1/n}}{2}\right)dx$. It follows from (1) that $\ln\dfrac{1+x^{1/n}}{2}\leqq\dfrac{1+x^{1/n}}{2}-1=\dfrac{x^{1/n}-1}{2}$.

$$\therefore I\leqq\int_1^2\frac{x^{1/n}-1}{2}dx=\frac{1}{2}\left[\frac{n}{n+1}x^{\frac{n+1}{n}}-x\right]_1^2$$

$$=\frac{1}{2}\left\{\frac{n}{n+1}\left(2^{\frac{n+1}{n}}-1\right)-1\right\}=\frac{n}{n+1}2^{1/n}-\frac{2n+1}{2(n+1)}.$$

$$\therefore nI\leqq\frac{n}{n+1}\left(n\cdot 2^{1/n}-\frac{2n+1}{2}\right)=\frac{n}{n+1}\left(\frac{2^{1/n}-1}{1/n}-\frac{1}{2}\right)①$$

Next, the graph of $u=g(t)=\ln t$ $(t>0)$ is upwards convex, because $g'(t)=\dfrac{1}{t}$ and $g''(t)=-\dfrac{1}{t^2}<0$.



Thus it lies over the segment connecting $(1,0)$ and $(a,\ln a)$, where $a>1$.

In other words, if $1\leqq t\leqq a$, $\ln t\geqq\dfrac{\ln a}{a-1}(t-1)$ …②

If $1\leqq x\leqq 2$, $1\leqq\dfrac{1+x^{1/n}}{2}\leqq\dfrac{1+2^{1/n}}{2}$.

Letting $t=\dfrac{1+x^{1/n}}{2}$ and $a=\dfrac{1+2^{1/n}}{2}$ in ②, we get

$$\ln\frac{1+x^{1/n}}{2}\geqq\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}\cdot\frac{x^{1/n}-1}{2}.$$

$$\therefore I\geqq\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}\int_1^2\frac{x^{\frac{1}{n}}-1}{2}dx$$

$$=\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}\left(\frac{n}{n+1}2^{\frac{1}{n}}-\frac{2n+1}{2(n+1)}\right).$$

$$\therefore nI\geqq\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}\cdot\frac{n}{n+1}\left(\frac{2^{1/n}-1}{1/n}-\frac{1}{2}\right)\ \cdots③$$

By the way, $\displaystyle\lim_{n\to\infty}\frac{n}{n+1}=1$,

$$\lim_{n\to\infty}\frac{2^{1/n}-1}{1/n}=\lim_{1/n\to0}\frac{2^{1/n}-2^0}{1/n-0}=\frac{d}{dt}2^t\Big|_{t=0}=2^t\ln2\Big|_{t=0}=\ln2.$$

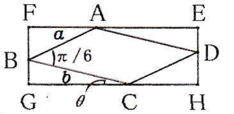
Hence the R.H.S. of ① converges to $\ln2-\dfrac{1}{2}$ as $n\to\infty$. Furthermore, $v:=2^{1/n}\to1$ as $n\to\infty$, and

$$\lim_{n\to\infty}\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}=\lim_{v\to1}\frac{\ln\frac{1+v}{2}}{\frac{v-1}{2}}=\lim_{v\to1}2\frac{\ln(1+v)-\ln2}{v-1}$$

$=2\dfrac{d}{dt}\ln(1+t)\Big|_{t=1}=2\cdot\dfrac{1}{1+t}\Big|_{t=1}=1$. Hence the R.H.S. of ③ converges to $\ln2-\dfrac{1}{2}$ as $n\to\infty$.

Thus we get $\displaystyle\lim_{n\to\infty}nI=\ln2-\frac{1}{2}$ by squeezing.

3 (1) As ABCD is a parallelogram, we have $\angle CDA=\angle ABC=\frac{\pi}{6}$,



$\angle BCD=\angle DAB=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$.

$CD=AB=a$, $DA=BC=b$.

AS $\angle BCG=\theta$, $\angle CBG=\frac{\pi}{2}-\theta$.

$\therefore ABF=\pi-\frac{\pi}{6}-(\frac{\pi}{2}-\theta)=\frac{\pi}{3}+\theta$.

$\therefore FG=FB+BG=a\cos(\theta+\frac{\pi}{3})+b\sin\theta$ …①

Similarly, $\angle DCH=\pi-\frac{5\pi}{6}-\theta=\frac{\pi}{6}-\theta$,

$GH=GC+CH=b\cos\theta+a\cos(\frac{\pi}{6}-\theta)$ …②

$\therefore S=FG\cdot GH$

$$\underset{\text{①,②}}{=}\{a\cos(\theta+\frac{\pi}{3})+b\sin\theta\}\{a\cos(\frac{\pi}{6}-\theta)+b\cos\theta\}$$

$$=\{a\sin(\frac{\pi}{6}-\theta)+b\sin\theta\}\{a\cos(\frac{\pi}{6}-\theta)+b\cos\theta\}$$

$$=a^2\sin(\frac{\pi}{6}-\theta)\cos(\frac{\pi}{6}-\theta)+b^2\sin\theta\cos\theta$$

$$+ab\{\sin(\frac{\pi}{6}-\theta)\cos\theta+\cos(\frac{\pi}{6}-\theta)\sin\theta\}$$

$$=\frac{a^2}{2}\sin(\frac{\pi}{3}-2\theta)+\frac{b^2}{2}\sin2\theta+ab\sin\frac{\pi}{6}$$

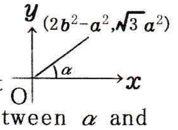
$$=\frac{a^2}{2}(\frac{\sqrt{3}}{2}\cos2\theta-\frac{1}{2}\sin2\theta)+\frac{b^2}{2}\sin2\theta+\frac{ab}{2}$$

$$=\frac{2b^2-a^2}{4}\sin2\theta+\frac{\sqrt{3}a^2}{4}\cos2\theta+\frac{ab}{2}.$$

(2) $\theta$ ranges between 0 and $\frac{\pi}{6}$, and it follows from (1) that

$$S=\frac{\sqrt{a^4-a^2b^2+b^4}}{2}\sin(2\theta+\alpha)+\frac{ab}{2},$$

where $\alpha$ is the angle indicated on the right.



As $0<a\leqq b$, $2b^2-a^2>0$ so that $\alpha$ is acute. $2\theta+\alpha$ ranges between $\alpha$ and $\frac{\pi}{3}+\alpha$. Furthermore, $\frac{\pi}{3}+\alpha\geqq\frac{\pi}{2}$ holds

iff $\alpha\geqq\frac{\pi}{6}\Leftrightarrow\tan\alpha\geqq\frac{1}{\sqrt{3}}\Leftrightarrow\frac{\sqrt{3}a^2}{2b^2-a^2}\geqq\frac{1}{\sqrt{3}}$

$\Leftrightarrow 3a^2\geqq2b^2-a^2\Leftrightarrow b\leqq\sqrt{2}a$.

Therefore, if $a\leqq b\leqq\sqrt{2}a$, $S$ takes maximum $\frac{\sqrt{a^4-a^2b^2+b^4}+ab}{2}$ when $\theta=\frac{\pi-2\alpha}{4}$, and if $b>\sqrt{2}a$, $S$ takes maximum $\frac{\sqrt{3}b^2+2ab}{4}$ when $\theta=\frac{\pi}{6}$.

4 (1) Proof by contradiction.

If $f_a(n)=n^2+n-a$ is a square number, there exists a nonnegative integer $m$ such that

$n^2+n-a=m^2$ …①

Suppose $n>a$. Then the L.H.S. of ① is greater than $n^2$ so that $m\geqq n+1$. Hence the R.H.S. of ① is not less than $(n+1)^2$, and

(R.H.S.)−(L.H.S.)$\geqq(n+1)^2-(n^2+n-a)=n+a+1>0$,

a contradiction.

(2) Consider a pair of integers $(n,m)$, where $n>0$, $m\geqq0$, satisfying

① $\Leftrightarrow (n+\frac{1}{2})^2-m^2=a+\frac{1}{4}$

$\Leftrightarrow (2n+2m+1)(2n-2m+1)=4a+1$ …①’

As $4a+1$ is an odd integer not less than 5, $2n+2m+1$, $2n-2m+1$ are positive odd integers satisfying $2n+2m+1\geqq2n-2m+1$ …②

(ii)⇒(i): If $4a+1$ is a prime, it follows from ①’ and ② that $\begin{cases}2n+2m+1=4a+1\\2n-2m+1=1\end{cases}\Leftrightarrow n=m=a$

$\therefore N_a=1$.

(i)⇒(ii): We shall prove the contrapositive.

If $4a+1$ is not a prime, we can express $4a+1=pq$, where $p,q$ are positive odd integers satisfying $3\leqq p\leqq q$.

$p,q$ are congruent to 1 or 3 modulo 4, and the chart on the right gives us $(p,q)\equiv(1,1)$ or $(3,3)$ modulo 4. Therefore,

| $p$ mod 4 | 1 | 1 | 3 | 3 |
|---|---|---|---|---|
| $q$ mod 4 | 1 | 3 | 1 | 3 |
| $pq$ mod 4 | 1 | 3 | 3 | 1 |

$\begin{cases}2n+2m+1=q\\2n-2m+1=p\end{cases}\Leftrightarrow (n,m)=(\frac{p+q-2}{4},\frac{q-p}{4})$

is a pair satisfying ①’ and different from $(a,a)$, i.e. $N_a\geqq2$.

Thus (i)⇔(ii) is proved.

5 Note that the card $n$ stays in the rightmost position after the 1st ($T_{n-1}$), and the card 1 stays in the leftmost position after the 2nd ($T_1$).

(1) Proof by contradiction.

If $\{A_1,A_2\}=\{k,\ell\}$ ($k>\ell>2$), then the arrangement after the 1st ($T_1$) is $\ell kA_3\cdots A_n$.The card $\ell$ stays there until the 2nd ($T_2$), and will be replaced by the card 1 at the 2nd ($T_1$). In other words, the 2nd card (from the left end) will end in $\ell$ (>2), a contradiction.

(2) It follows from (1) that

$\{A_1,A_2\}=\{1,2\}$‥①, $\{1,k\}$‥②, or $\{2,\ell\}$‥③,

where $k,\ell$ are integers not less than 3.

Case①. $(A_1,A_2)=(1,2)$ or $(2,1)$, and the 1st $(T_1)$ makes the arrangement $12A_3\cdots A_n$.
The cards 1 and 2 stay there, and $A_3\cdots A_n$ is reordered to $3\cdots n$ after the $2(n-3)$ operations. Thus the number of possible arrangement $A_3\cdots A_n$ is $C_{n-2}$.

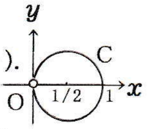
Case②. $(A_1,A_2)=(1,k)$ or $(k,1)$, and the 1st $(T_1)$ makes the arrangement $1kA_3\cdots A_n$.
The card 1 stays there, and $kA_3\cdots A_n$ is reordered to $2\cdots n$ after the $2(n-2)$ operations. Thus the number of possible arrangement $kA_3\cdots A_n$ is $\underset{\substack{\uparrow\\ \text{leftmost card is 2}}}{C_{n-1}}-C_{n-2}$.

Case③. $(A_1,A_2)=(2,\ell)$ or $(\ell,2)$, and the 1st $(T_1)$ makes the arrangement $2\ell A_3\cdots A_n$.
The card 2 stays there until the 2nd $(T_2)$, and will be replaced by the card 1 at the 2nd $(T_1)$. The number of possible arrangement $\ell A_3\cdots A_n$ is $\underset{\substack{\uparrow\\ \text{leftmost card is 1}}}{C_{n-1}}-C_{n-2}$ as in the case②.

Finally,
$$C_n=\underset{①}{2C_{n-2}}+\underset{②}{2(C_{n-1}-C_{n-2})}+\underset{③}{2(C_{n-1}-C_{n-2})}$$
$$=4C_{n-1}-2C_{n-2}.$$

Remark Solving the recurrence $C_2=2$, $C_3=6$, and $C_n=4C_{n-1}-2C_{n-2}$ $(n\geqq 4)$, we get $C_n=\dfrac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$ $(n\geqq 2)$.

6 (1) As $z\in C$, we can express $z=\dfrac{1}{2}(1+\cos\theta)+\dfrac{i}{2}\sin\theta$ $(-\pi<\theta<\pi)$.

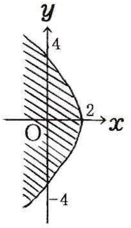


Then
$$\frac{1}{z}=\frac{2}{(1+\cos\theta)+i\sin\theta}=\frac{1}{\cos\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2}+i\sin\dfrac{\theta}{2}\right)}$$
$$=\frac{1}{\cos\dfrac{\theta}{2}}\left(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}\right)=1-i\tan\frac{\theta}{2}.$$
$\therefore \boldsymbol{Re}\left(\dfrac{1}{z}\right)=1$.

(2) If $\alpha,\beta\in C$, (1) allows us to express $\dfrac{1}{\alpha}=1+ai$, $\dfrac{1}{\beta}=1+bi$, where $a,b\in \boldsymbol{R}$.
Furtheremore, $-\pi<\theta<\pi$ implies that $-\tan\dfrac{\theta}{2}$ can be any real number. In other words, $a,b$ range the entire reals provided $a\neq b$.
$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=1+2ai-a^2+1+2bi-b^2=(2-a^2-b^2)+2(a+b)i$$

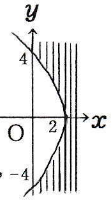
If we put $(x,y)=(2-a^2-b^2,2(a+b))$,
$$ab=\frac{(a+b)^2-(a^2+b^2)}{2}=\frac{\left(\dfrac{y}{2}\right)^2-(2-x)}{2}=\frac{4x+y^2}{8}-1.$$
Hence $a,b$ are the distinct real roots of the quadratic equation $t^2-\dfrac{y}{2}t+\left(\dfrac{4x+y^2}{8}-1\right)=0$, whose discriminant $\dfrac{y^2}{4}-4\left(\dfrac{4x+y^2}{8}-1\right)>0$ $\Leftrightarrow x<2-\dfrac{y^2}{8}$.

Therefore, $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$ moves around the domain on the right (boundary excluded).



(3) If we put $\gamma=x+yi$, where $x,y$ are reals, then it follows from (2) that $\overline{x}\geqq 2-\dfrac{y^2}{8}$ …①

Furthermore, $\dfrac{1}{\gamma}=\dfrac{\overline{\gamma}}{\gamma\overline{\gamma}}=\dfrac{x-yi}{x^2+y^2}$ gives us $\boldsymbol{Re}\left(\dfrac{1}{\gamma}\right)=\dfrac{x}{x^2+y^2}$.



If there exists a maximum of $\dfrac{x}{x^2+y^2}$, it happens when $x>0$. If we fix a positive $x$,
$$\begin{cases} y \text{ moves around the entire reals if } x\geqq 2,\\ y^2\geqq 16-8x\ (\because ①) \text{ if } 0<x<2.\end{cases}$$
As $\boldsymbol{Re}\left(\dfrac{1}{\gamma}\right)$ is decreasing in $y^2$, it takes
$$\text{maximum}\begin{cases}\dfrac{1}{x} \text{ at } y=0, \text{ if } x\geqq 2,\\[1ex] \dfrac{x}{(x-4)^2} \text{ at } y=\pm 2\sqrt{4-2x}, \text{ if } 0<x<2.\end{cases}$$
Hence the maximum $M$ of $\boldsymbol{Re}\left(\dfrac{1}{\gamma}\right)$ is
$$\max\left\{\frac{1}{2},\ \max_{0<x<2}\frac{x}{(x-4)^2}\right\}.$$
Similarly, the minimum $m$ of $\boldsymbol{Re}\left(\dfrac{1}{\gamma}\right)$ is
$$\min_{x<0}\frac{x}{(x-4)^2}.$$
If we put $f(x)=\dfrac{x}{(x-4)^2}$ $(x<2)$,
$f'(x)=\dfrac{(x-4)^2-x\cdot 2(x-4)}{(x-4)^4}=\dfrac{x+4}{(4-x)^3}$ gives us the chart on the right.

| $x$ | | $-4$ | | $(2)$ |
|---|---|---|---|---|
| $f'(x)$ | $-$ | $0$ | $+$ | |
| $f(x)$ | ↘ | $-\dfrac{1}{16}$ | ↗ | $\left(\dfrac{1}{2}\right)$ |

Therefore, $M=\dfrac{1}{2}$ $(\gamma=2)$, $m=-\dfrac{1}{16}$ $(\gamma=-4\pm 4\sqrt{3}i)$

# Math Exam given for the 2025 general admission process at Univ. of Tokyo

## Problems for those who wish to major in Literature, Economics, etc. (100 min.)

$\boxed{1}$ Let $a$ be a positive number, and C: $y=x^2$ be a parabola on a coordinate plane. Let $\ell$ be the normal to C at P$(a,a^2)$ (the line orthogonal to the tangent to C at P), and Q be the point of intersection of $\ell$ and C, other than P.

(1) Find the $x$-coordinate of Q.

Let $m$ be the normal to C at Q, and R be the point of intersection of $m$ and C, other than Q.

(2) Find the minimum of the $x$-coordinate of R, if $a$ varies in the set of positive numbers.

$\boxed{2}$ Consider an isosceles triangle ABC satisfying AB=AC=1 on a plane. Let $r>0$, and $D_r$ be the union of 3 circles of radius $r$ centered at A,B and C, where all the triangle and circles consist of their circumferences and insides. Let $s$ be the minimal $r$ such that $D_r$ contains all the edges AB, AC and BC, and $t$ be the minimal $r$ such that $D_r$ contains $\triangle$ABC.

(1) Find the $s$ and $t$, if $\angle\mathrm{BAC}=\dfrac{\pi}{3}$.

(2) Find the $s$ and $t$, if $\angle\mathrm{BAC}=\dfrac{2\pi}{3}$.

(3) If $\angle\mathrm{BAC}=\theta$, where $0<\theta<\pi$, express the $s$ and $t$ in terms of $\theta$.

$\boxed{3}$ 2 white balls are put side by side. Tossing a coin whose head and tail will show up equally, add a white ball or a black one in the row according to the process below.

Process (*): Add a white (black) ball at the right end of the row if the head (tail) of the coin shows up. Furthermore, if the rightmost 3 balls turn WBW (BWB), where W (B) means a white (black) ball, replace them by WWW (BBB).

For example, if we toss the coin twice and the tail and head show up in order, the row will be of 4 white balls.

Let $n$ be a positive integer and consider the row of $(n+2)$ balls after applying the process (*) $n$ times.

(1) If $n=3$, find the probability that the 2nd ball from the right end is white.

(2) Let $n$ be a positive integer. Find the probability that the 2nd ball from the right end is white.

(3) Let $n$ be a positive integer. Find the probability that the rightmost 2 balls are white.

$\boxed{4}$ Let $a$ be a real number and S$(a)$ be the area of the region defined by $\begin{cases} y\leqq -\dfrac{1}{2}x^2+2 \\ y\geqq |x^2+a| \\ -1\leqq x\leqq 1 \end{cases}$ on a coordinate plane.

Find the maximum of S$(a)$, if $a$ moves over the interval $-2\leqq a<2$.

# Solution for the 2025 Math Exam at Univ. of Tokyo (Literature, etc.)

$\boxed{1}$ (1) As $(x^2)'=2x$, $2a$ is the slope of the tangent to C at $\mathrm{P}(a,a^2)$. Hence $\ell$ is the line passing through P with slope $-\frac{1}{2a}$:

$$y=-\frac{1}{2a}(x-a)+a^2=-\frac{1}{2a}x+\frac{1}{2}+a^2.$$

The $x$-coordinate of a point of intersection of C and $\ell$ is a root of $x^2=-\frac{1}{2a}x+\frac{1}{2}+a^2$

$$\Leftrightarrow x^2+\frac{1}{2a}x-\frac{1}{2}-a^2=(x-a)(x+a+\frac{1}{2a})=0$$

$$\Leftrightarrow x=a,-a-\frac{1}{2a}.$$

Therefore, the $x$-coordinate of Q is $-a-\frac{1}{2a}$.

(2) Letting $b=-a-\frac{1}{2a}$ $(<0)$, we get

$m$: $y=-\frac{1}{2b}x+\frac{1}{2}+b^2$ as we did in (1).

Also, the $x$-coordinate of R is $x_{\mathrm{R}}=-b-\frac{1}{2b}$.

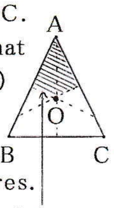
By the way, $a+\frac{1}{2a}\geqq 2\sqrt{a\cdot\frac{1}{2a}}=\sqrt{2}$, where the equality holds when $a=\frac{1}{2a}\Leftrightarrow a=\frac{1}{\sqrt{2}}$, which follows from $a>0$ and the fact that "the arithmetic mean of 2 positive numbers is not less than their geometric mean."

Hence $b=-(a+\frac{1}{2a})\leqq-\sqrt{2}$, and the equality holds when $a=\frac{\sqrt{2}}{2}$.

Furthermore, $\frac{dx_{\mathrm{R}}}{db}=-1+\frac{1}{2b^2}=\frac{1-2b^2}{2b^2}$ so that $\frac{dx_{\mathrm{R}}}{db}<0$ and $x_{\mathrm{R}}$ is monotone decreasing over the interval $b\leqq-\sqrt{2}$.

Therefore, $x_{\mathrm{R}}$ takes the minimum $\frac{5\sqrt{2}}{4}$ at $b=-\sqrt{2}\Leftrightarrow a=\frac{\sqrt{2}}{2}$.

$\boxed{2}$ First note that any point of $\mathrm{D}_r$ belongs to at least one of the 3 circles of radius $r$, centered at A,B and C. Also, the minimal $r$ such that $\mathrm{P}\in\mathrm{D}_r$ is the minimum of PA,PB and PC. Moreover, the set of points P such that PA=PB (PB=PC, PC=PA, respectively) is the perpendicular bisector of the edge AB (BC,CA, respectively), and expressed as a dotted line in the figures.



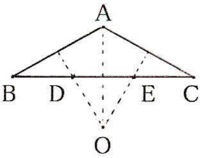
The set of points whose nearest vertex is A, among which the circumcenter O is the farthest from A.

(1) If $\angle\mathrm{BAC}=\frac{\pi}{3}$, $\triangle\mathrm{ABC}$ is equilateral.

Among the points on $\partial(\triangle\mathrm{ABC})$, the circumference of $\triangle\mathrm{ABC}$, the midpoints of the edges are the farthest from their nearest vertices. Thus $s=\frac{1}{2}$.

Among the points of $\triangle\mathrm{ABC}$, the circumcenter is the farthest from its nearest vertices. Hence the sine rules gives us

$$t=\frac{1}{2\sin\frac{\pi}{3}}=\frac{\sqrt{3}}{3}.$$



(2) If $\angle\mathrm{BAC}=\frac{2\pi}{3}$, $\angle\mathrm{ABC}=\angle\mathrm{ACB}=\frac{\pi}{6}$, and the points D,E shown are the farthest from their nearest vertices, among the points of $\triangle\mathrm{ABC}$.

$\therefore s=t=\mathrm{BD}$ $(=\mathrm{AD}=\mathrm{AE}=\mathrm{CE})$.

As $\mathrm{BD}\cos\frac{\pi}{6}=\frac{\mathrm{AB}}{2}=\frac{1}{2}$, $\mathrm{BD}=\frac{1}{2}\cdot\frac{2}{\sqrt{3}}=\frac{\sqrt{3}}{3}$.

$\therefore s=t=\frac{\sqrt{3}}{3}$.

(3) If $\angle\mathrm{BAC}=\theta$, $\angle\mathrm{ABC}=\angle\mathrm{ACB}=\frac{\pi-\theta}{2}$.

(i) If $0<\theta<\frac{\pi}{3}$, $\angle\mathrm{ABC}=\angle\mathrm{ACB}>\angle\mathrm{BAC}$ so that $\mathrm{AB}=\mathrm{AC}>\mathrm{BC}$. Therefore, among the points of $\partial(\triangle\mathrm{ABC})$, the midpoints of AB,AC are the farthest from their nearest vertices. $\therefore s=\frac{1}{2}$.

Among the points of $\triangle\mathrm{ABC}$, the circumcenter is the farthest from its nearest vertices so that the sine rule gives us

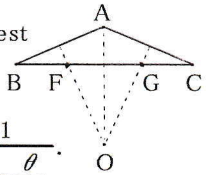
$$t=\frac{\mathrm{AB}}{2\sin\angle\mathrm{ACB}}=\frac{1}{2\sin(\frac{\pi}{2}-\frac{\theta}{2})}=\frac{1}{2\cos\frac{\theta}{2}}.$$

(ii) If $\frac{\pi}{3}<\theta\leqq\frac{\pi}{2}$, $\angle\mathrm{BAC}>\angle\mathrm{ABC}=\angle\mathrm{ACB}$ so that $\mathrm{BC}>\mathrm{AB}=\mathrm{AC}$. Therefore, among the points of $\partial(\triangle\mathrm{ABC})$, the midpoint of BC is the farthest from its nearest vertices.

$$\therefore s=\frac{\mathrm{BC}}{2}=\mathrm{AB}\cos(\frac{\pi}{2}-\frac{\theta}{2})=\sin\frac{\theta}{2}.$$

Quite similarly as in (i), $t=\frac{1}{2\cos\frac{\theta}{2}}$.

(iii) If $\frac{\pi}{2}<\theta<\pi$, the points F,G shown right are the farthest from their nearest vertices, among the points of $\triangle\mathrm{ABC}$.



$$\therefore s=t=\mathrm{BF}=\frac{\mathrm{AB}/2}{\cos(\frac{\pi}{2}-\frac{\theta}{2})}=\frac{1}{2\sin\frac{\theta}{2}}.$$

It follows from (1) and (i)-(iii) that

$$s=\begin{cases}\dfrac{1}{2} & (0<\theta\leqq\dfrac{\pi}{3})\\ \sin\dfrac{\theta}{2} & (\dfrac{\pi}{3}<\theta\leqq\dfrac{\pi}{2}),\\ \dfrac{1}{2\sin\dfrac{\theta}{2}} & (\dfrac{\pi}{2}<\theta<\pi)\end{cases}$$

$$t=\begin{cases}\dfrac{1}{2\cos\dfrac{\theta}{2}} & (0<\theta\leqq\dfrac{\pi}{2})\\ \dfrac{1}{2\sin\dfrac{\theta}{2}} & (\dfrac{\pi}{2}<\theta<\pi)\end{cases}$$

$\boxed{3}$ (1) We will denote the head (tail) of the coin by H (T).

If $n=3$, the following 8 cases can happen equally:

| 1st toss | 2nd toss | 3rd toss | the row of balls |
|---|---|---|---|
| H | H | H | WWWWWW |
| H | H | T | WWWWWB |
| H | T | H | WWWWWW |
| H | T | T | WWWWBB |
| T | H | H | WWWWWW |
| T | H | T | WWWWWB |
| T | T | H | WWBBBW |
| T | T | T | WWBBBB |

Therefore, the probability to find is $\dfrac{5}{8}$.

(2)(3) Let $p_n$ ($q_n$) be the probability that the 2nd ball from the right end is white (the rightmost 2 balls are white) after applying the process (*) $n$ times.

The rightmost 2 balls in the row are (i) WW, (ii) WB, (iii) BW or (iv) BB, and let $a_n, b_n, c_n, d_n$ be the probability that (i),(ii),(iii),(iv) happens after applying the process (*) $n$ times. Then

$p_n=a_n+b_n\cdots$①, $q_n=a_n\cdots$②, $a_n+b_n+c_n+d_n=1\cdots$③,

and $a_1=b_1=\dfrac{1}{2}$, $c_1=d_1=0\cdots$④

Furthermore, the rightmost 2 balls of the row after the $(n+1)$th process will be:

| (n+1)th toss<br>the rightmost 2 balls after the nth process | H | T |
|---|---|---|
| (i) | WW | WB |
| (ii) | WW | BB |
| (iii) | WW | BB |
| (iv) | BW | BB |

Hence $a_{n+1}=\dfrac{1}{2}(a_n+b_n+c_n)\cdots$⑤, $b_{n+1}=\dfrac{1}{2}a_n\cdots$⑥

$c_{n+1}=\dfrac{1}{2}d_n\cdots$⑦, $d_{n+1}=\dfrac{1}{2}(b_n+c_n+d_n)$

③,⑤ and ⑦ give us $a_{n+1}+c_{n+1}=\dfrac{1}{2}$.

This and ④ imply that $c_n=\dfrac{1}{2}-a_n\cdots$⑧ for $n\geqq1$.

⑤ and ⑧ give us $a_{n+1}=\dfrac{1}{2}(b_n+\dfrac{1}{2})\cdots$⑨

⑥+⑨: $a_{n+1}+b_{n+1}=\dfrac{1}{2}(a_n+b_n)+\dfrac{1}{4}$

$\underset{①}{\Leftrightarrow}\ p_{n+1}=\dfrac{1}{2}p_n+\dfrac{1}{4}\ \Leftrightarrow\ p_{n+1}-\dfrac{1}{2}=\dfrac{1}{2}(p_n-\dfrac{1}{2})$.

This and ①,④ tell us that $\{p_n-\dfrac{1}{2}\}$ is a geometric sequence with initial term $p_1-\dfrac{1}{2}=\dfrac{1}{2}$ and the common ratio $\dfrac{1}{2}$.

$\therefore p_n-\dfrac{1}{2}=(\dfrac{1}{2})^n\ \Leftrightarrow\ \underline{p_n=\dfrac{1}{2}+(\dfrac{1}{2})^n}\cdots$⑩

solution for (2)

Next, ⑨−⑥: $a_{n+1}-b_{n+1}=-\dfrac{1}{2}(a_n-b_n)+\dfrac{1}{4}$

$\Leftrightarrow\ a_{n+1}-b_{n+1}-\dfrac{1}{6}=-\dfrac{1}{2}(a_n-b_n-\dfrac{1}{6})$.

This and ④ tell us that $\{a_n-b_n-\dfrac{1}{6}\}$ is a geometric sequence with initial term $a_1-b_1-\dfrac{1}{6}=-\dfrac{1}{6}$ and the common ratio $-\dfrac{1}{2}$.

$\therefore a_n-b_n-\dfrac{1}{6}=-\dfrac{1}{6}(-\dfrac{1}{2})^{n-1}\Leftrightarrow a_n-b_n=\dfrac{1}{6}+\dfrac{1}{3}(-\dfrac{1}{2})^n$

This and ⑩ $\underset{①}{\Leftrightarrow}$ $a_n+b_n=\dfrac{1}{2}+(\dfrac{1}{2})^n$ give us

$2a_n=\dfrac{2}{3}+(\dfrac{1}{2})^n+\dfrac{1}{3}(-\dfrac{1}{2})^n$

$\underset{②}{\Leftrightarrow}\ \underline{q_n=\dfrac{1}{3}+\dfrac{1}{2^{n+1}}+\dfrac{1}{6}(-\dfrac{1}{2})^n}$

solution for (3)

$\boxed{4}$

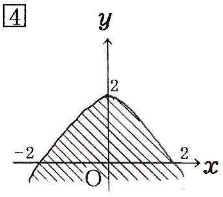


Fig.1

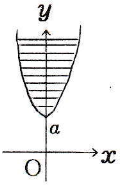


Fig.2

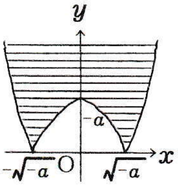


Fig.3

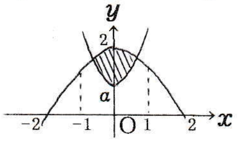
$y \leqq -\frac{1}{2}x^2+2$ is expressed as in the Fig.1, and $y \geqq |x^2+a|$ is expressed as in the Fig.2(3) if $a \geqq 0$ ($a<0$). All of them are symmetric with respect to the $y$-axis.

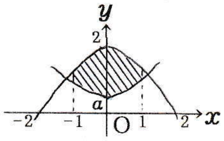
Furthermore, $y=-\frac{1}{2}x^2+2$ and $y=x^2+a$ intersect in $x=\pm\sqrt{\frac{2}{3}(2-a)}$, and

$-\frac{1}{2}x^2+2-(-x^2-a)=\frac{x^2}{2}+2+a \geqq 0$ if $-2 \leqq a<0$.

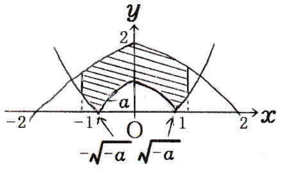
Hence the domain to consider is as follows:

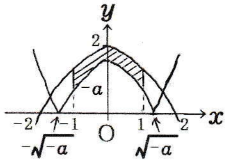
(i) $\frac{2}{3}(2-a) \leqq 1$, i.e. $\frac{1}{2} \leqq a<2$ (ii) $0 \leqq a<\frac{1}{2}$





(iii) $0<\sqrt{-a}<1$, i.e. $-1<a<0$ (iv) $-2 \leqq a \leqq -1$





Case(i) $S(a)=\int_{-\sqrt{\frac{2}{3}(2-a)}}^{\sqrt{\frac{2}{3}(2-a)}}\{-\frac{1}{2}x^2+2-(x^2+a)\}dx$

$=\frac{3}{2}\cdot\frac{1}{6}(2\sqrt{\frac{2}{3}(2-a)})^3=2(\sqrt{\frac{2}{3}(2-a)})^3$,

which takes the maximum 2 at $a=\frac{1}{2}$.

Case(ii) $S(a)=\int_{-1}^{1}\{-\frac{1}{2}x^2+2-(x^2+a)\}dx$

$=2\int_0^1(-\frac{3}{2}x^2+2-a)dx=[-x^3+(4-2a)x]_0^1=3-2a$,

which takes the maximum 3 at $a=0$.

Case(iii) $S(a)=2\int_0^{\sqrt{-a}}\{-\frac{1}{2}x^2+2-(-x^2-a)\}dx$

$+2\int_{\sqrt{-a}}^{1}\{-\frac{1}{2}x^2+2-(x^2+a)\}dx$

$=[\frac{x^3}{3}+(4+2a)x]_0^{\sqrt{-a}}+[-x^3+(4-2a)x]_{\sqrt{-a}}^1$

$=\frac{8}{3}a\sqrt{-a}-2a+3$

If we put $b=\sqrt{-a}$, $0<b<1$ and $S(a)=-\frac{8}{3}b^3+2b^2+3$.

$\therefore \frac{d}{db}S(a)=-8b^2+4b=4b(1-2b)$

| $b$ | (0) | | $\frac{1}{2}$ | | (1) |
|---|---|---|---|---|---|
| $\frac{d}{db}S(a)$ | | + | 0 | − | |
| $S(a)$ | | ↗ | | ↘ | |

Thus we get the table on the right, and see that $S(a)$ takes the maximum $\frac{19}{6}$ when $b=\frac{1}{2} \Leftrightarrow a=-\frac{1}{4}$.

Case(iv) $S(a)=2\int_0^1\{-\frac{1}{2}x^2+2-(-x^2-a)\}dx$

$=[\frac{x^3}{3}+(4+2a)x]_0^1=2a+\frac{13}{3}$,

which takes the maximum $\frac{7}{3}$ at $a=-1$.

It follows from (i)-(iv) that $S(a)$ takes the maximum $\frac{19}{6}$ at $a=-\frac{1}{4}$.

<u>Remark</u> The grapf of $c=S(a)$.