Math Exam given for the 2025 general admission process at Univ. of Tokyo

Problems for those who wish to major in Science, Engineering, etc. (150 min.)

[1] Let A(0,0), B(0,1), C(1,1), D(1,0) be points on a coordinate plane. Let t satisfy $0<t<1$, P_t, Q_t, R_t be the points on the segments AB,BC,CD, respectively, such that $\frac{AP_t}{P_tB}=\frac{BQ_t}{Q_tC}=\frac{CR_t}{R_tD}=\frac{t}{1-t}$, S_t, T_t be the points on the segments P_tQ_t, Q_tR_t, respectively, such that $\frac{P_tS_t}{S_tQ_t}=\frac{Q_tT_t}{T_tR_t}=\frac{t}{1-t}$, and U_t be the point on the segment S_tT_t such that $\frac{S_tU_t}{U_tT_t}=\frac{t}{1-t}$.
Furthermore, let A,D be U_0, U_1, respectively.

(1) Find the coordinates of the point U_t.

(2) Find the area of the domain surrounded by the segment AD and the curve traced by the point U_t, $0 \leqq t \leqq 1$.

(3) Let a satisfy $0<a<1$. Express the length of the curve traced by the point U_t, $0 \leqq t \leqq a$, as a polynomial in a.

[2] (1) Prove $\ln x \leqq x-1$ for $x>0$. (2) Find $\lim_{n\to\infty} n\int_1^2 \ln\left(\frac{1+x^{\frac{1}{n}}}{2}\right)dx$.

[3] A parallelogram ABCD satisfies $\angle ABC=\frac{\pi}{6}$, AB=a, BC=b, and $a \leqq b$.
Consider a rectangle with the condition:
The vertices A,B,C,D lie on the edges EF,FG,GH,HE, respectively, where an edge includes its ends.
Let S be the area of the rectangle EFGH.

(1) Express S in terms of a,b and $\theta=\angle BCG$.

(2) Express the maximum of S in terms of a and b.

[4] A square number is the square of a nonnegative integer.
Let a be a positive integer, and $f_a(x)=x^2+x-a$.

(1) Let n be a positive integer. Prove that $n \leqq a$, if $f_a(n)$ is a square number.

(2) Denote by N_a the number of positive integers n such that $f_a(n)$ is a square number.
Prove that the conditions (i),(ii) below are equivalent:
(i) $N_a=1$ (ii) $4a+1$ is a prime.

[5] There're $n(\geqq 2)$ cards numbered 1 through n, and we arrange them in a row.
Consider the following operation (T_i), where $i=1,2,\cdots$, or $n-1$.
(T_i) If the number of the ith card (from the left end) is greater than that of the $(i+1)$th one, we switch these 2 cards. Otherwise, we do nothing.
Suppose that the number of the ith card is A_i $(1 \leqq i \leqq n)$ in the beginning, and it turns i for $i=1,\cdots,n$ by $(n-1)$ operations $(T_1),(T_2),\cdots,(T_{n-1})$ followed by $(n-1)$ operations $(T_{n-1}),\cdots,(T_2),(T_1)$.

(1) Prove that at least one of A_1, A_2 is not greater than 2.

(2) Let C_n be the number of possible arrangement $A_1 \cdots A_n$.
For $n \geqq 4$, express C_n in terms of C_{n-1} and C_{n-2}.

[6] On a plane of complex numbers, let C be the circle centered at $\frac{1}{2}$ with radius $\frac{1}{2}$, minus zero.

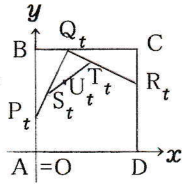
(1) For $z \in C$, prove that the real part of $\frac{1}{z}$ is 1.

(2) If $\alpha, \beta \in C$ and they're distinct, express the domain in which $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ moves around.

(3) If γ is a complex number belonging to the complement of the domain in (2), find the maximum and the minimum of the real part of $\frac{1}{\gamma}$.

Solution for the 2025 Math Exam at Univ. of Tokyo (Science, etc.)

[1] (1) First, we get $P_t(0,t)$, $Q_t(t,1)$, $R_t(1,1-t)$. Next,



$\overrightarrow{OS_t}=(1-t)\overrightarrow{OP_t}+t\overrightarrow{OQ_t}$

$=(0,t(1-t))+(t^2,t)=(t^2,2t-t^2)$,

$\overrightarrow{OT_t}=(1-t)\overrightarrow{OQ_t}+t\overrightarrow{OR_t}$

$=(t(1-t),1-t)+(t,t(1-t))=(2t-t^2,1-t^2)$,

$\overrightarrow{OU_t}=(1-t)\overrightarrow{OS_t}+t\overrightarrow{OT_t}$

$=(t^2(1-t),(1-t)(2t-t^2))+(t(2t-t^2),t(1-t^2))$

$=(3t^2-2t^3,3t-3t^2)$. $\therefore U_t(3t^2-2t^3,3t-3t^2)$.

(2) Let $(x,y)=(3t^2-2t^3,3t-3t^2)$, $0\leqq t\leqq 1$.

$\frac{dx}{dt}=6t-6t^2=6t(1-t)$,

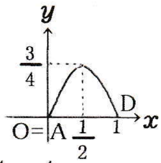
$\frac{dy}{dt}=3-6t$ give us the table and the graph on the right.

t	0		$\frac{1}{2}$		1
$\frac{dx}{dt}$		$+$		$+$	
$\frac{dy}{dt}$		$+$	0	$-$	
$\binom{x}{y}$	$\binom{0}{0}$	↗	$\binom{1/2}{3/4}$	↘	$\binom{1}{0}$

Thus the area to find is $S=\int_0^1 y\,dx$.

If we put $x=3t^2-2t^3$, we get $y=3t-3t^2$, $dx=(6t-6t^2)dt$, and

x	$0\to 1$
t	$0\to 1$



$\therefore S=\int_0^1(3t-3t^2)(6t-6t^2)dt$

$=18\int_0^1(t^2-2t^3+t^4)dt=18[\frac{t^3}{3}-\frac{t^4}{2}+\frac{t^4}{5}]_0^1=\frac{3}{5}$.

(3) The length of the curve is

$\int_0^a\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt$

$=\int_0^a\sqrt{36t^2(1-t)^2+(3-6t)^2}\,dt$

$=3\int_0^a\sqrt{4t^4-8t^3+8t^2-4t+1}\,dt$

$=3\int_0^a(2t^2-2t+1)dt$ ($\because 2t^2-2t+1=2(t-\frac{1}{2})^2+\frac{1}{2}>0$)

$=[2t^3-3t^2+3t]_0^a=2a^3-3a^2+3a$.

[2] (1) Let $f(x)=\ln x-(x-1)$, $x>0$.

Then $f'(x)=\frac{1}{x}-1$ gives the table on the right. Hence $f(x)\leqq f(1)=0$, i.e. $\ln x\leqq x-1$.

x	(0)		1	
$f'(x)$		$+$	0	$-$
$f(x)$		↗		↘

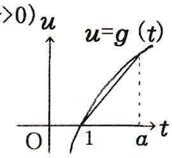
(2) Let $I=\int_1^2\ln(\frac{1+x^{1/n}}{2})dx$. It follows from (1) that $\ln\frac{1+x^{1/n}}{2}\leqq\frac{1+x^{1/n}}{2}-1=\frac{x^{1/n}-1}{2}$.

$\therefore I\leqq\int_1^2\frac{x^{1/n}-1}{2}dx=\frac{1}{2}[\frac{n}{n+1}x^{\frac{n+1}{n}}-x]_1^2$

$=\frac{1}{2}\{\frac{n}{n+1}(2^{\frac{n+1}{n}}-1)-1\}=\frac{n}{n+1}2^{1/n}-\frac{2n+1}{2(n+1)}$.

$\therefore nI\leqq\frac{n}{n+1}(n\cdot 2^{1/n}-\frac{2n+1}{2})=\frac{n}{n+1}(\frac{2^{1/n}-1}{1/n}-\frac{1}{2})$ ①

Next, the graph of $u=g(t)=\ln t$ $(t>0)$ is upwards convex, because $g'(t)=\frac{1}{t}$ and $g''(t)=-\frac{1}{t^2}<0$.



Thus it lies over the segment connecting $(1,0)$ and $(a,\ln a)$, where $a>1$.

In other words, if $1\leqq t\leqq a$, $\ln t\geqq\frac{\ln a}{a-1}(t-1)$ ⋯②

If $1\leqq x\leqq 2$, $1\leqq\frac{1+x^{1/n}}{2}\leqq\frac{1+2^{1/n}}{2}$.

Letting $t=\frac{1+x^{1/n}}{2}$ and $a=\frac{1+2^{1/n}}{2}$ in ②, we get

$\ln\frac{1+x^{1/n}}{2}\geqq\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}\cdot\frac{x^{1/n}-1}{2}$.

$\therefore I\geqq\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}\int_1^2\frac{x^{\frac{1}{n}}-1}{2}dx$

$=\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}(\frac{n}{n+1}2^{\frac{1}{n}}-\frac{2n+1}{2(n+1)})$.

$\therefore nI\geqq\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}\cdot\frac{n}{n+1}(\frac{2^{1/n}-1}{1/n}-\frac{1}{2})$ ⋯③

By the way, $\lim_{n\to\infty}\frac{n}{n+1}=1$,

$\lim_{n\to\infty}\frac{2^{1/n}-1}{1/n}=\lim_{1/n\to 0}\frac{2^{1/n}-2^0}{1/n-0}=\frac{d}{dt}2^t|_{t=0}=2^t\ln 2|_{t=0}=\ln 2$.

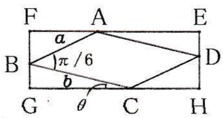
Hence the R.H.S. of ① converges to $\ln 2-\frac{1}{2}$ as $n\to\infty$. Furthermore, $v:=2^{1/n}\to 1$ as $n\to\infty$, and

$\lim_{n\to\infty}\frac{\ln\frac{1+2^{1/n}}{2}}{\frac{2^{1/n}-1}{2}}=\lim_{v\to 1}\frac{\ln\frac{1+v}{2}}{\frac{v-1}{2}}=\lim_{v\to 1}2\frac{\ln(1+v)-\ln 2}{v-1}$

$=2\frac{d}{dt}\ln(1+t)|_{t=1}=2\cdot\frac{1}{1+t}|_{t=1}=1$. Hence the R.H.S. of ③ converges to $\ln 2-\frac{1}{2}$ as $n\to\infty$.

Thus we get $\lim_{n\to\infty}nI=\ln 2-\frac{1}{2}$ by squeezing.

[3] (1) As ABCD is a parallelogram, we have $\angle CDA=\angle ABC=\frac{\pi}{6}$,



$\angle BCD=\angle DAB=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$.

$CD=AB=a$, $DA=BC=b$.

AS $\angle BCG=\theta$, $\angle CBG=\frac{\pi}{2}-\theta$.

$\therefore ABF=\pi-\frac{\pi}{6}-(\frac{\pi}{2}-\theta)=\frac{\pi}{3}+\theta$.

$\therefore FG=FB+BG=a\cos(\theta+\frac{\pi}{3})+b\sin\theta$ ···①

Similarly, $\angle DCH=\pi-\frac{5\pi}{6}-\theta=\frac{\pi}{6}-\theta$,

$GH=GC+CH=b\cos\theta+a\cos(\frac{\pi}{6}-\theta)$ ···②

$\therefore S=FG\cdot GH$

$\underset{①,②}{=}\{a\cos(\theta+\frac{\pi}{3})+b\sin\theta\}\{a\cos(\frac{\pi}{6}-\theta)+b\cos\theta\}$

$=\{a\sin(\frac{\pi}{6}-\theta)+b\sin\theta\}\{a\cos(\frac{\pi}{6}-\theta)+b\cos\theta\}$

$=a^2\sin(\frac{\pi}{6}-\theta)\cos(\frac{\pi}{6}-\theta)+b^2\sin\theta\cos\theta$

$\quad+ab\{\sin(\frac{\pi}{6}-\theta)\cos\theta+\cos(\frac{\pi}{6}-\theta)\sin\theta\}$

$=\frac{a^2}{2}\sin(\frac{\pi}{3}-2\theta)+\frac{b^2}{2}\sin 2\theta+ab\sin\frac{\pi}{6}$

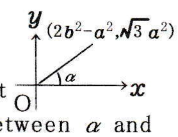
$=\frac{a^2}{2}(\frac{\sqrt{3}}{2}\cos 2\theta-\frac{1}{2}\sin 2\theta)+\frac{b^2}{2}\sin 2\theta+\frac{ab}{2}$

$=\frac{2b^2-a^2}{4}\sin 2\theta+\frac{\sqrt{3}a^2}{4}\cos 2\theta+\frac{ab}{2}$.

(2) θ ranges between 0 and $\frac{\pi}{6}$, and it follows from (1) that

$S=\frac{\sqrt{a^4-a^2b^2+b^4}}{2}\sin(2\theta+\alpha)+\frac{ab}{2}$,

where α is the angle indicated on the right.



As $0<a\leqq b$, $2b^2-a^2>0$ so that α is acute. $2\theta+\alpha$ ranges between α and $\frac{\pi}{3}+\alpha$. Furthermore, $\frac{\pi}{3}+\alpha\geqq\frac{\pi}{2}$ holds iff $\alpha\geqq\frac{\pi}{6}\Leftrightarrow\tan\alpha\geqq\frac{1}{\sqrt{3}}\Leftrightarrow\frac{\sqrt{3}a^2}{2b^2-a^2}\geqq\frac{1}{\sqrt{3}}$

$\Leftrightarrow 3a^2\geqq 2b^2-a^2\Leftrightarrow b\leqq\sqrt{2}a$.

Therefore, if $a\leqq b\leqq\sqrt{2}a$, S takes maximum $\frac{\sqrt{a^4-a^2b^2+b^4}+ab}{2}$ when $\theta=\frac{\pi-2\alpha}{4}$, and if $b>\sqrt{2}a$, S takes maximum $\frac{\sqrt{3}b^2+2ab}{4}$ when $\theta=\frac{\pi}{6}$.

[4] (1) Proof by contradiction.

If $f_a(n)=n^2+n-a$ is a square number, there exists a nonnegative integer m such that

$n^2+n-a=m^2$ ···①

Suppose $n>a$. Then the L.H.S. of ① is greater than n^2 so that $m\geqq n+1$. Hence the R.H.S. of ① is not less than $(n+1)^2$, and

(R.H.S.)−(L.H.S.)$\geqq(n+1)^2-(n^2+n-a)=n+a+1>0$,

a contradiction.

(2) Consider a pair of integers (n,m), where $n>0$, $m\geqq 0$, satisfying

① $\Leftrightarrow (n+\frac{1}{2})^2-m^2=a+\frac{1}{4}$

$\Leftrightarrow (2n+2m+1)(2n-2m+1)=4a+1$ ···①'

As $4a+1$ is an odd integer not less than 5, $2n+2m+1$, $2n-2m+1$ are positive odd integers satisfying $2n+2m+1\geqq 2n-2m+1$ ···②

(ii)⇒(i): If $4a+1$ is a prime, it follows from ①' and ② that $\begin{cases}2n+2m+1=4a+1\\2n-2m+1=1\end{cases}\Leftrightarrow n=m=a$

$\therefore N_a=1$.

(i)⇒(ii): We shall prove the contrapositive.

If $4a+1$ is not a prime, we can express $4a+1=pq$, where p,q are positive odd integers satisfying $3\leqq p\leqq q$.

p,q are congruent to 1 or 3 modulo 4, and the chart on the right gives us $(p,q)\equiv(1,1)$ or $(3,3)$ modulo 4. Therefore,

p mod 4	1	1	3	3
q mod 4	1	3	1	3
pq mod 4	1	3	3	1

$\begin{cases}2n+2m+1=q\\2n-2m+1=p\end{cases}\Leftrightarrow (n,m)=(\frac{p+q-2}{4},\frac{q-p}{4})$

is a pair satisfying ①' and different from (a,a), i.e. $N_a\geqq 2$.

Thus (i)⇔(ii) is proved.

[5] Note that the card n stays in the rightmost position after the 1st (T_{n-1}), and the card 1 stays in the leftmost position after the 2nd (T_1).

(1) Proof by contradiction.

If $\{A_1,A_2\}=\{k,\ell\}$ ($k>\ell>2$), then the arrangement after the 1st (T_1) is $\ell k A_3\cdots A_n$. The card ℓ stays there until the 2nd (T_2), and will be replaced by the card 1 at the 2nd (T_1). In other words, the 2nd card (from the left end) will end in ℓ (>2), a contradiction.

(2) It follows from (1) that

$\{A_1,A_2\}=\{1,2\}$··①, $\{1,k\}$··②, or $\{2,\ell\}$··③,

where k,ℓ are integers not less than 3.

Case①. $(A_1,A_2)=(1,2)$ or $(2,1)$, and the 1st (T_1) makes the arrangement $12A_3\cdots A_n$.
The cards 1 and 2 stay there, and $A_3\cdots A_n$ is reordered to $3\cdots n$ after the $2(n-3)$ operations. Thus the number of possible arrangement $A_3\cdots A_n$ is C_{n-2}.

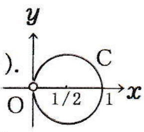
Case②. $(A_1,A_2)=(1,k)$ or $(k,1)$, and the 1st (T_1) makes the arrangement $1kA_3\cdots A_n$.
The card 1 stays there, and $kA_3\cdots A_n$ is reordered to $2\cdots n$ after the $2(n-2)$ operations. Thus the number of possible arrangement $kA_3\cdots A_n$ is $C_{n-1}-C_{n-2}$.
↑ leftmost card is 2

Case③. $(A_1,A_2)=(2,\ell)$ or $(\ell,2)$, and the 1st (T_1) makes the arrangement $2\ell A_3\cdots A_n$.
The card 2 stays there until the 2nd (T_2), and will be replaced by the card 1 at the 2nd (T_1). The number of possible arrangement $\ell A_3\cdots A_n$ is $C_{n-1}-C_{n-2}$ as in the case②.
↑ leftmost card is 1

Finally, $C_n=\underset{①}{2C_{n-2}}+\underset{②}{2(C_{n-1}-C_{n-2})}+\underset{③}{2(C_{n-1}-C_{n-2})}$
$=4C_{n-1}-2C_{n-2}$.

Remark Solving the recurrence $C_2=2$, $C_3=6$, and $C_n=4C_{n-1}-2C_{n-2}$ $(n\geqq 4)$, we get $C_n=\dfrac{(2+\sqrt{2})^{n-1}+(2-\sqrt{2})^{n-1}}{2}$ $(n\geqq 2)$.

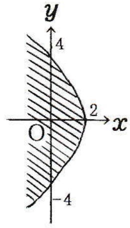
6 (1) As $z\in C$, we can express $z=\frac{1}{2}(1+\cos\theta)+\frac{i}{2}\sin\theta$ $(-\pi<\theta<\pi)$.



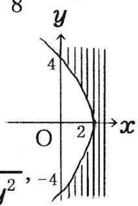
Then
$$\frac{1}{z}=\frac{2}{(1+\cos\theta)+i\sin\theta}=\frac{1}{\cos\frac{\theta}{2}(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2})}$$
$$=\frac{1}{\cos\frac{\theta}{2}}(\cos\frac{\theta}{2}-i\sin\frac{\theta}{2})=1-i\tan\frac{\theta}{2}.$$
$\therefore Re\,(\frac{1}{z})=1$.

(2) If $\alpha,\beta\in C$, (1) allows us to express $\frac{1}{\alpha}=1+ai$, $\frac{1}{\beta}=1+bi$, where $a,b\in R$.
Furtheremore, $-\pi<\theta<\pi$ implies that $-\tan\frac{\theta}{2}$ can be any real number. In other words, a,b range the entire reals provided $a\neq b$.
$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=1+2ai-a^2+1+2bi-b^2=(2-a^2-b^2)+2(a+b)i$$
If we put $(x,y)=(2-a^2-b^2,2(a+b))$,
$$ab=\frac{(a+b)^2-(a^2+b^2)}{2}=\frac{(\frac{y}{2})^2-(2-x)}{2}=\frac{4x+y^2}{8}-1.$$
Hence a,b are the distinct real roots of the quadratic equation $t^2-\frac{y}{2}t+(\frac{4x+y^2}{8}-1)=0$, whose discriminant $\frac{y^2}{4}-4(\frac{4x+y^2}{8}-1)>0$
$\Leftrightarrow x<2-\frac{y^2}{8}$.

Therefore, $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ moves around the domain on the right (boundary excluded).



(3) If we put $\gamma=x+yi$, where x,y are reals, then it follows from (2) that $x\geqq 2-\frac{y^2}{8}$ …①
Furthermore, $\frac{1}{\gamma}=\frac{\bar{\gamma}}{\gamma\bar{\gamma}}=\frac{x-yi}{x^2+y^2}$ gives us $Re(\frac{1}{\gamma})=\frac{x}{x^2+y^2}$.



If there exists a maximum of $\frac{x}{x^2+y^2}$, it happens when $x>0$. If we fix a positive x,
$\begin{cases} y \text{ moves around the entire reals if } x\geqq 2, \\ y^2\geqq 16-8x\ (\because ①) \text{ if } 0<x<2. \end{cases}$
As $Re(\frac{1}{\gamma})$ is decreasing in y^2, it takes
$$\text{maximum}\begin{cases}\frac{1}{x} \text{ at } y=0, \text{ if } x\geqq 2, \\ \frac{x}{(x-4)^2} \text{ at } y=\pm 2\sqrt{4-2x}, \text{ if } 0<x<2.\end{cases}$$
Hence the maximum M of $Re(\frac{1}{\gamma})$ is
$$\max\{\frac{1}{2}, \max_{0<x<2}\frac{x}{(x-4)^2}\}.$$
Similarly, the minimum m of $Re(\frac{1}{\gamma})$ is
$$\min_{x<0}\frac{x}{(x-4)^2}.$$
If we put $f(x)=\frac{x}{(x-4)^2}$ $(x<2)$,
$f'(x)=\frac{(x-4)^2-x\cdot 2(x-4)}{(x-4)^4}=\frac{x+4}{(4-x)^3}$ gives us the chart on the right.

x		-4		(2)
$f'(x)$	$-$	0	$+$	
$f(x)$	↘	$-\frac{1}{16}$	↗	$(\frac{1}{2})$

Therefore, $M=\frac{1}{2}$ $(\gamma=2)$,
$m=-\frac{1}{16}$ $(\gamma=-4\pm 4\sqrt{3}i)$

Math Exam given for the 2025 general admission process at Univ. of Tokyo

Problems for those who wish to major in Literature, Economics, etc. (100 min.)

1 Let a be a positive number, and C: $y=x^2$ be a parabola on a coordinate plane. Let ℓ be the normal to C at P(a,a^2) (the line orthogonal to the tangent to C at P), and Q be the point of intersection of ℓ and C, other than P.

(1) Find the x-coordinate of Q.

Let m be the normal to C at Q, and R be the point of intersection of m and C, other than Q.

(2) Find the minimum of the x-coordinate of R, if a varies in the set of positive numbers.

2 Consider an isosceles triangle ABC satisfying AB=AC=1 on a plane. Let $r>0$, and D_r be the union of 3 circles of radius r centered at A,B and C, where all the triangle and circles consist of their circumferences and insides. Let s be the minimal r such that D_r contains all the edges AB, AC and BC, and t be the minimal r such that D_r contains △ABC.

(1) Find the s and t, if $\angle BAC=\frac{\pi}{3}$.

(2) Find the s and t, if $\angle BAC=\frac{2\pi}{3}$.

(3) If $\angle BAC=\theta$, where $0<\theta<\pi$, express the s and t in terms of θ.

3 2 white balls are put side by side. Tossing a coin whose head and tail will show up equally, add a white ball or a black one in the row according to the process below.

Process (*): Add a white (black) ball at the right end of the row if the head (tail) of the coin shows up. Furthermore, if the rightmost 3 balls turn WBW (BWB), where W (B) means a white (black) ball, replace them by WWW (BBB).

For example, if we toss the coin twice and the tail and head show up in order, the row will be of 4 white balls.

Let n be a positive integer and consider the row of $(n+2)$ balls after applying the process (*) n times.

(1) If $n=3$, find the probability that the 2nd ball from the right end is white.

(2) Let n be a positive integer. Find the probability that the 2nd ball from the right end is white.

(3) Let n be a positive integer. Find the probability that the rightmost 2 balls are white.

4 Let a be a real number and S(a) be the area of the region defined by $\begin{cases} y \leqq -\frac{1}{2}x^2+2 \\ y \geqq |x^2+a| \\ -1 \leqq x \leqq 1 \end{cases}$ on a coordinate plane.

Find the maximum of S(a), if a moves over the interval $-2 \leqq a < 2$.

Solution for the 2025 Math Exam at Univ. of Tokyo (Literature, etc.)

1 (1) As $(x^2)'=2x$, $2a$ is the slope of the tangent to C at $P(a,a^2)$. Hence ℓ is the line passing through P with slope $-\dfrac{1}{2a}$:

$$y=-\frac{1}{2a}(x-a)+a^2=-\frac{1}{2a}x+\frac{1}{2}+a^2.$$

The x-coordinate of a point of intersection of C and ℓ is a root of $x^2=-\dfrac{1}{2a}x+\dfrac{1}{2}+a^2$

$$\Leftrightarrow x^2+\frac{1}{2a}x-\frac{1}{2}-a^2=(x-a)(x+a+\frac{1}{2a})=0$$

$$\Leftrightarrow x=a,-a-\frac{1}{2a}.$$

Therefore, the x-coordinate of Q is $-a-\dfrac{1}{2a}$.

(2) Letting $b=-a-\dfrac{1}{2a}$ (<0), we get

m: $y=-\dfrac{1}{2b}x+\dfrac{1}{2}+b^2$ as we did in (1).

Also, the x-coordinate of R is $x_R=-b-\dfrac{1}{2b}$.

By the way, $a+\dfrac{1}{2a}\geqq 2\sqrt{a\cdot\dfrac{1}{2a}}=\sqrt{2}$, where the equality holds when $a=\dfrac{1}{2a}\Leftrightarrow a=\dfrac{1}{\sqrt{2}}$, which follows from $a>0$ and the fact that "the arithmetic mean of 2 positive numbers is not less than their geometric mean."

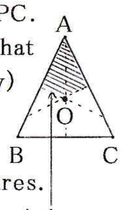
Hence $b=-(a+\dfrac{1}{2a})\leqq-\sqrt{2}$, and the equality holds when $a=\dfrac{\sqrt{2}}{2}$.

Furthermore, $\dfrac{dx_R}{db}=-1+\dfrac{1}{2b^2}=\dfrac{1-2b^2}{2b^2}$ so that $\dfrac{dx_R}{db}<0$ and x_R is monotone decreasing over the interval $b\leqq-\sqrt{2}$.

Therefore, x_R takes the minimum $\dfrac{5\sqrt{2}}{4}$ at $b=-\sqrt{2}\Leftrightarrow a=\dfrac{\sqrt{2}}{2}$.

2 First note that any point of D_r belongs to at least one of the 3 circles of radius r, centered at A,B and C. Also, the minimal r such that $P\in D_r$ is the minimum of PA,PB and PC.

Moreover, the set of points P such that PA=PB (PB=PC, PC=PA, respectively) is the perpendicular bisector of the edge AB (BC,CA, respectively), and expressed as a dotted line in the figures.



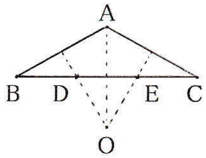
The set of points whose nearest vertex is A, among which the circumcenter O is the farthest from A.

(1) If $\angle BAC=\dfrac{\pi}{3}$, $\triangle ABC$ is equilateral.

Among the points on $\partial(\triangle ABC)$, the circumference of $\triangle ABC$, the midpoints of the edges are the farthest from their nearest vertices. Thus $s=\dfrac{1}{2}$.

Among the points of $\triangle ABC$, the circumcenter is the farthest from its nearest vertices. Hence the sine rules gives us

$$t=\frac{1}{2\sin\frac{\pi}{3}}=\frac{\sqrt{3}}{3}.$$



(2) If $\angle BAC=\dfrac{2\pi}{3}$, $\angle ABC=\angle ACB=\dfrac{\pi}{6}$, and the points D,E shown are the farthest from their nearest vertices, among the points of $\triangle ABC$.

$\therefore s=t=BD$ $(=AD=AE=CE)$.

As $BD\cos\dfrac{\pi}{6}=\dfrac{AB}{2}=\dfrac{1}{2}$, $BD=\dfrac{1}{2}\cdot\dfrac{2}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}$.

$\therefore s=t=\dfrac{\sqrt{3}}{3}$.

(3) If $\angle BAC=\theta$, $\angle ABC=\angle ACB=\dfrac{\pi-\theta}{2}$.

(i) If $0<\theta<\dfrac{\pi}{3}$, $\angle ABC=\angle ACB>\angle BAC$ so that $AB=AC>BC$. Therefore, among the points of $\partial(\triangle ABC)$, the midpoints of AB,AC are the farthest from their nearest vertices. $\therefore s=\dfrac{1}{2}$.

Among the points of $\triangle ABC$, the circumcenter is the farthest from its nearest vertices so that the sine rule gives us

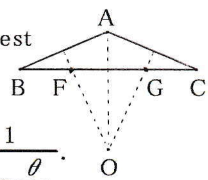
$$t=\frac{AB}{2\sin\angle ACB}=\frac{1}{2\sin(\frac{\pi}{2}-\frac{\theta}{2})}=\frac{1}{2\cos\frac{\theta}{2}}.$$

(ii) If $\dfrac{\pi}{3}<\theta\leqq\dfrac{\pi}{2}$, $\angle BAC>\angle ABC=\angle ACB$ so that $BC>AB=AC$. Therefore, among the points of $\partial(\triangle ABC)$, the midpoint of BC is the farthest from its nearest vertices.

$$\therefore s=\frac{BC}{2}=AB\cos(\frac{\pi}{2}-\frac{\theta}{2})=\sin\frac{\theta}{2}.$$

Quite similarly as in (i), $t=\dfrac{1}{2\cos\frac{\theta}{2}}$.

(iii) If $\dfrac{\pi}{2}<\theta<\pi$, the points F,G shown right are the farthest from their nearest vertices, among the points of $\triangle ABC$.



$$\therefore s=t=BF=\frac{AB/2}{\cos(\frac{\pi}{2}-\frac{\theta}{2})}=\frac{1}{2\sin\frac{\theta}{2}}.$$

It follows from (1) and (i)–(iii) that

$$s=\begin{cases}\dfrac{1}{2} & (0<\theta\leqq\dfrac{\pi}{3})\\ \sin\dfrac{\theta}{2} & (\dfrac{\pi}{3}<\theta\leqq\dfrac{\pi}{2})\\ \dfrac{1}{2\sin\dfrac{\theta}{2}} & (\dfrac{\pi}{2}<\theta<\pi)\end{cases},$$

$$t=\begin{cases}\dfrac{1}{2\cos\dfrac{\theta}{2}} & (0<\theta\leqq\dfrac{\pi}{2})\\ \dfrac{1}{2\sin\dfrac{\theta}{2}} & (\dfrac{\pi}{2}<\theta<\pi)\end{cases}$$

3 (1) We will denote the head (tail) of the coin by H (T).

If $n=3$, the following 8 cases can happen equally:

1st toss	2nd toss	3rd toss	the row of balls
H	H	H	WWWWWW
H	H	T	WWWWWB
H	T	H	WWWWWW
H	T	T	WWWWBB
T	H	H	WWWWWW
T	H	T	WWWWWB
T	T	H	WWBBW
T	T	T	WWBBB

Therefore, the probability to find is $\frac{5}{8}$.

(2)(3) Let p_n (q_n) be the probability that the 2nd ball from the right end is white (the rightmost 2 balls are white) after applying the process (*) n times.

The rightmost 2 balls in the row are (i) WW, (ii) WB, (iii) BW or (iv) BB, and let a_n, b_n, c_n, d_n be the probability that (i),(ii),(iii),(iv) happens after applying the process (*) n times. Then

$p_n=a_n+b_n$ ⋯①, $q_n=a_n$ ⋯②, $a_n+b_n+c_n+d_n=1$ ⋯③, and $a_1=b_1=\frac{1}{2}$, $c_1=d_1=0$ ⋯④

Furthermore, the rightmost 2 balls of the row after the $(n+1)$th process will be:

the rightmost 2 balls after the nth process \ $(n+1)$th toss	H	T
(i)	WW	WB
(ii)	WW	BB
(iii)	WW	BB
(iv)	BW	BB

Hence $a_{n+1}=\frac{1}{2}(a_n+b_n+c_n)$ ⋯⑤, $b_{n+1}=\frac{1}{2}a_n$ ⋯⑥

$c_{n+1}=\frac{1}{2}d_n$ ⋯⑦, $d_{n+1}=\frac{1}{2}(b_n+c_n+d_n)$

③,⑤ and ⑦ give us $a_{n+1}+c_{n+1}=\frac{1}{2}$.

This and ④ imply that $c_n=\frac{1}{2}-a_n$ ⋯⑧ for $n\geqq 1$.

⑤ and ⑧ give us $a_{n+1}=\frac{1}{2}(b_n+\frac{1}{2})$ ⋯⑨

⑥+⑨: $a_{n+1}+b_{n+1}=\frac{1}{2}(a_n+b_n)+\frac{1}{4}$

$\underset{①}{\Leftrightarrow} p_{n+1}=\frac{1}{2}p_n+\frac{1}{4} \Leftrightarrow p_{n+1}-\frac{1}{2}=\frac{1}{2}(p_n-\frac{1}{2})$.

This and ①,④ tell us that $\{p_n-\frac{1}{2}\}$ is a geometric sequence with initial term $p_1-\frac{1}{2}=\frac{1}{2}$ and the common ratio $\frac{1}{2}$.

$\therefore p_n-\frac{1}{2}=(\frac{1}{2})^n \Leftrightarrow \underline{p_n=\frac{1}{2}+(\frac{1}{2})^n}$ ⋯⑩

solution for (2)

Next, ⑨−⑥: $a_{n+1}-b_{n+1}=-\frac{1}{2}(a_n-b_n)+\frac{1}{4}$

$\Leftrightarrow a_{n+1}-b_{n+1}-\frac{1}{6}=-\frac{1}{2}(a_n-b_n-\frac{1}{6})$.

This and ④ tell us that $\{a_n-b_n-\frac{1}{6}\}$ is a geometric sequence with initial term $a_1-b_1-\frac{1}{6}=-\frac{1}{6}$ and the common ratio $-\frac{1}{2}$.

$\therefore a_n-b_n-\frac{1}{6}=-\frac{1}{6}(-\frac{1}{2})^{n-1} \Leftrightarrow a_n-b_n=\frac{1}{6}+\frac{1}{3}(-\frac{1}{2})^n$

This and ⑩ $\underset{①}{\Leftrightarrow} a_n+b_n=\frac{1}{2}+(\frac{1}{2})^n$ give us

$2a_n=\frac{2}{3}+(\frac{1}{2})^n+\frac{1}{3}(-\frac{1}{2})^n$

$\underset{②}{\Leftrightarrow} \underline{q_n=\frac{1}{3}+\frac{1}{2^{n+1}}+\frac{1}{6}(-\frac{1}{2})^n}$

solution for (3)

4

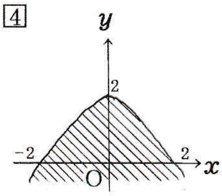


Fig.1

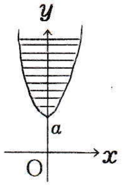


Fig.2

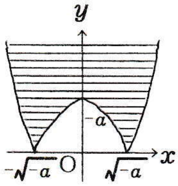


Fig.3

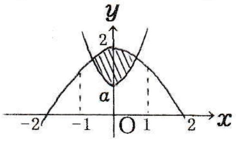
$y \leqq -\dfrac{1}{2}x^2+2$ is expressed as in the Fig.1, and $y \geqq |x^2+a|$ is expressed as in the Fig.2(3) if $a \geqq 0$ $(a<0)$. All of them are symmetric with respect to the y-axis.

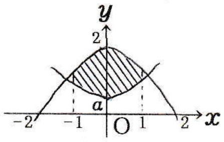
Furthermore, $y=-\dfrac{1}{2}x^2+2$ and $y=x^2+a$ intersect in $x=\pm\sqrt{\dfrac{2}{3}(2-a)}$, and

$-\dfrac{1}{2}x^2+2-(-x^2-a)=\dfrac{x^2}{2}+2+a \geqq 0$ if $-2 \leqq a<0$.

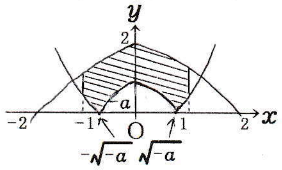
Hence the domain to consider is as follows:

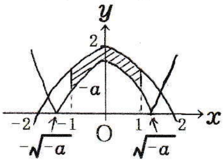
(i) $\dfrac{2}{3}(2-a) \leqq 1$, i.e. $\dfrac{1}{2} \leqq a<2$ (ii) $0 \leqq a<\dfrac{1}{2}$





(iii) $0<\sqrt{-a}<1$, i.e. $-1<a<0$ (iv) $-2 \leqq a \leqq -1$





Case(i) $S(a)=\displaystyle\int_{-\sqrt{\frac{2}{3}(2-a)}}^{\sqrt{\frac{2}{3}(2-a)}}\{-\frac{1}{2}x^2+2-(x^2+a)\}dx$

$=\dfrac{3}{2}\cdot\dfrac{1}{6}\left(2\sqrt{\dfrac{2}{3}(2-a)}\right)^3=2\left(\sqrt{\dfrac{2}{3}(2-a)}\right)^3$,

which takes the maximum 2 at $a=\dfrac{1}{2}$.

Case(ii) $S(a)=\displaystyle\int_{-1}^{1}\{-\frac{1}{2}x^2+2-(x^2+a)\}dx$

$=2\displaystyle\int_0^1\left(-\frac{3}{2}x^2+2-a\right)dx=[-x^3+(4-2a)x]_0^1=3-2a$,

which takes the maximum 3 at $a=0$.

Case(iii) $S(a)=2\displaystyle\int_0^{\sqrt{-a}}\{-\frac{1}{2}x^2+2-(-x^2-a)\}dx$

$+2\displaystyle\int_{\sqrt{-a}}^{1}\{-\frac{1}{2}x^2+2-(x^2+a)\}dx$

$=\left[\dfrac{x^3}{3}+(4+2a)x\right]_0^{\sqrt{-a}}+[-x^3+(4-2a)x]_{\sqrt{-a}}^1$

$=\dfrac{8}{3}a\sqrt{-a}-2a+3$

If we put $b=\sqrt{-a}$, $0<b<1$ and $S(a)=-\dfrac{8}{3}b^3+2b^2+3$.

$\therefore \dfrac{d}{db}S(a)=-8b^2+4b=4b(1-2b)$

Thus we get the table on the right, and see that $S(a)$ takes the maximum $\dfrac{19}{6}$ when $b=\dfrac{1}{2} \Leftrightarrow a=-\dfrac{1}{4}$.

b	(0)		$\frac{1}{2}$		(1)
$\frac{d}{db}S(a)$		+	0	-	
$S(a)$		↗		↘	

Case(iv) $S(a)=2\displaystyle\int_0^1\{-\frac{1}{2}x^2+2-(-x^2-a)\}dx$

$=\left[\dfrac{x^3}{3}+(4+2a)x\right]_0^1=2a+\dfrac{13}{3}$,

which takes the maximum $\dfrac{7}{3}$ at $a=-1$.

It follows from (i)-(iv) that $S(a)$ takes the maximum $\dfrac{19}{6}$ at $a=-\dfrac{1}{4}$.

Remark The grapf of $c=S(a)$.